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1917

OPHTHALMIC LENSES AND PRISMS

An Essay Contributed to "The American Encyclopedia of Ophthalmology"

BY

CHARLES F. PRENTICE, M. E.,

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FOREWORD

Through the courtesy of Dr. Casey A. Wood, Editor in Chief of *The American Encyclopedia of Ophthalmology,* the author of this essay has been able to secure from its publishers two hundred reprints for distribution among his professional friends and others thought to be vocationally interested in ophthalmic lenses and prisms; wherefore, this copy, containing the author's autographed portrait, is presented to

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Singer Building,
New York, 1917.

OPHTHALMIC LENSES AND PRISMS

FOR THE CONVENIENCE OF THE READER THE FOLLOWING LIST OF SUBHEADINGS HAS BEEN INDEXED FOR THIS IMPORTANT AND EXTENSIVE SECTION. THEY OCCUR IN THE ORDER OF THEIR DISCUSSION [ED.]:

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The Function of Ophthalmic Lenses.

The purpose of this essay is to describe the physical characteristics of ophthalmic lenses and prisms, and to lay before the student of ophthalmology at least those fundamental principles of their refraction with which every efficient eye-practitioner should be familiar; especially as the adaptation of lenses to vision is founded upon that commensurate knowledge of theoretic optics which embraces the measurable phenomena of light acted upon by mirrors, lenses and prisms. In other words, this unique branch of applied science is a special department of physics with which even the university student is not generally made familiar.

For this reason the various forms and properties of lenses used

exclusively to counteract optical anomalies of vision are here dealt with, and whose function it is to change only the direction of rays of light so that they may, after passing through the artificial medium, enter the optically anomalous eye in the precise direction for which it is structurally adapted in a state of rest to produce a well-defined retinal image within the physiologically normal eye-ball.

Since, even without artificial aid, all symmetrically formed *healthy* eyes, in this state of repose, are capable of producing clearly defined retinal images of objects that under suitable conditions emit or reflect

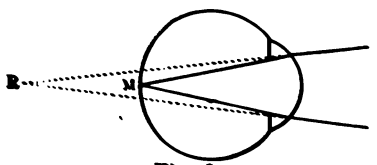

Fig. 1.
The Static Hyperopic Eye Adapted to Convergent Rays.

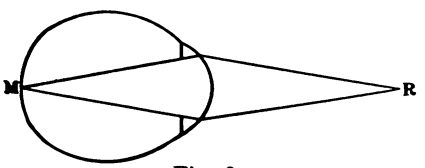

Fig. 3.
The Static Myopic Eye Adapted to Divergent Rays.

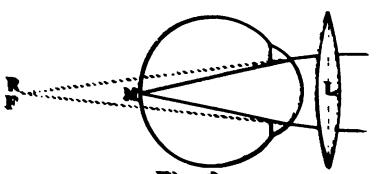

Fig. 2.
The Static Hyperopic Eye Lenticularly Corrected for Parallel Rays.

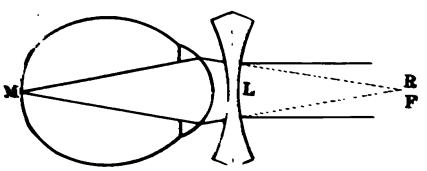

Fig. 4.
The Static Myopic Eye Lenticularly Corrected for Parallel Rays.

The Focus of the Artificial Lens is at the Farpoint of the Ametropic Eye.

either parallel, convergent or divergent rays, it has been found convenient to conventionally select the eye that is reposefully adapted to parallel rays as the normal or *emmetropic* eye; wherefore, eyes that are likewise statically adapted to convergent or divergent rays are merely considered optically anomalous or *ametropic* with respect to parallel incident rays of light. The difference between the refractive powers of the emmetropic and the ametropic eye is, therefore, equal to the power of the lens, L, required to cause parallel incident rays of light to assume convergence toward the farpoint (*punctum remotum*), R, *behind* the hyperopic eye, Fig. 2, or apparent divergence from the farpoint, R, *in front* of the myopic eye, Fig. 4, because each of these unaided eyes, Fig. 1 and Fig. 3, respectively, is statically adapted to focus upon the retina (macular center), M, incident light converging toward or diverging from the same reciprocal point, R. In short, the punctum remotum, R, and the focus, F, of the lens, L, are coincident points.

Therefore, ophthalmic lenses are of two kinds, namely, those which bend or *refract* parallel incident rays through convergence to a *real focus,* F, behind the lens, or through divergence from an apparent or *virtual focus,* F, in front of the lens, L, which may be made of any refracting medium more dense than air, usually glass, whose opposite surfaces, with respect to a common axis, are, in general, portions of intersecting spheric surfaces of like or unequal curvatures, often resembling a lentil in form, and from which at least the convex lens appropriately derived its name.

In short, a lens is usually a centered system of two refracting spheric surfaces, which causes light-rays to converge or diverge systematically after passing through it. In fact, in order that a refracting medium shall be consistently called a lens it must have the property of producing either convergence or divergence of rays of light, toward a real or from a virtual focal point, F,—or focal line, as with the cylindric lens. The effect of convergence produced by a lens is to create a *real image* of the object, and of divergence, only to figuratively form an image, which is then called a *virtual image.*

For this reason a thin glass having either flat or slightly curved parallel surfaces, even though mounted in spectacles, is not a lens. Flat glasses are called plano-glasses; those with curved parallel surfaces being known by the French name, "Coquilles" (shells), which may be parts of a blown sphere of glass, or made of glass whose concentrically curved surfaces are more perfectly produced through grinding. Such vitreous agents are generally made of colored glass, being used to modify the intensity of the light entering the eye. In earlier times they were made of blue or green glass, later of *smoke* glass—which is neutral in its effect upon the natural color of objects seen through it—and more recently, in various shades of violet, green, yellow and orange glass. The two latter colors absorb, more or less, the irritating violet or chemical rays, though not as effectually as Crookes' glass, having a refractive index of 1.523, which is now produced in two shades, designated as Absorptions A (light) and B (dark). The former, A, is a very pale, neutral tint which is almost colorless to the observer seeing it worn by others. It is said to transmit 99 per cent. of visible light and to absorb the harmful ultra-violet rays and 40 per cent. of the red (heat) rays, while objects viewed through it retain their natural colors the same as shade B, which is equally as effective with respect to the harmful rays, but designed to transmit only 84 per cent. of visible light. Various kinds of greenish-yellow glass, also claimed to suppress the ultra-violet rays to a superior degree, are being exploited

as Euphos, Noviol, Fieuzal, Chlorophyl and Hallour. Amber and Amethyst of various shades are also used.

Ophthalmic lenses are usually made of colorless glass, although the aforesaid colored glasses are sometimes advantageously applied; whereas, rock crystal, or so-called Brazilian pebble, has become almost obsolete for the purpose on account of its lack of homogeneity.

The Bausch & Lomb Optical Co. recently published some interesting information respecting the properties of optical glass from which, owing to its commendable brevity, the following is abstracted:

"Glass is an amorphous, transparent or translucent mixture of silicates by definite chemical formulæ. The essential materials for glass-making are silica, an alkali and lime or lead. Part of the lime or lead may be replaced by oxides of other metals, also by certain borates and phosphates to replace a part of the silica, especially in glass manufactured for optical purposes.

"Before the era of modern optical glass manufacture there were only two types of glass available to the optician, the one a lime glass with a low refraction and small dispersion (crown-glass), and the other a lead glass with a relatively high index and large dispersion (flint-glass). Although the terms "crown" and "flint" are still used, they have no definite meaning to the optician of today, as in late years the glass manufacturers, in answer to the demand of modern optics, have succeeded in putting upon the market so many new varieties that there is no longer the sharp division between the two kinds. It may be interesting to state here that in the last catalog of Schott & Genossen of Jena, Germany, there are listed *one hundred* different varieties of optical glass.

"The requirements demanded of the glass for ophthalmic lenses, while stringent, are not so many as those required for glass to be used in other optical instruments. Glass for ophthalmic lenses must be hard, durable, homogeneous, free from bubbles and striæ but also of a constant index of refraction. With the introduction of the deeper forms of lenses the moulding of glass has been introduced. This very materially shortens the subsequently applied process of grinding.

"Roentgen glass is a transparent glass which the Roentgen or X-ray can not penetrate. Lenses produced from this material offer protection to the patient, physician or operator while exposures with the X-ray are being made.

"Colored glass is produced by adding metal oxides to the mass of melted material: cobalt-oxide giving the blue, chrom-oxide, green; gold-oxide, ruby; silver-oxide, yellow, manganese-oxide, violet. Smoke glass is produced by using several of the above-mentioned oxides.

"Light when passing through glass will, generally speaking, suffer a reduction in intensity. The light lost in transmission is said to be absorbed, and the extent thereof is the intrinsic absorption of the glass. The volume of the light transmitted is said to be the intrinsic transmission of the glass. It has been satisfactorily proven that lost light is changed to heat.

"All glasses do not have the same absorption for the same color, nor does any one have the same absorption for all colors. Thus one specimen might pass a great deal of red and green and absorb nearly all blue light, while another might pass a great deal of green and blue and absorb nearly all the red light of the spectrum. Such absorption of particular colors or groups of colors is called *selective absorption.* A glass which absorbs equal proportions of colors of light is called a *neutral glass.* It serves the purpose of reducing the intensity of light without altering its color.

"Since absorbed light is entirely lost, the color of a piece of glass will be determined by the colors of light transmitted by it. If one looks at an object through a colored glass and the object gives off light only of the colors absorbed by the glass and in the proportions absorbed, the object will appear black. If the object gives off light only of colors transmitted by the glass and in the same proportions as transmitted, there will be no change in the color of the object. If the object gives off light, some of which is absorbed and some of which is transmitted by the glass, the object will seem to have only the colors transmitted by the glass.

"If for any reason it is desired to suppress any particular color or colors of light in proportions, it is necessary only to pass the light through a glass, the intrinsic absorption of which is in keeping with the requirements. We have prepared charts showing the relative intrinsic absorption of visible light of the colored glasses commonly used for ophthalmic lenses. The height of the rectangle represents 100 per cent., and the vertical lines indicate the various colors of the spectrum—red, yellow, green, blue and violet. The portions of the vertical lines below the curve indicate absorbed light, and those above indicate transmitted light. Thus, for example, the absorptions for Euphos A are:

	Per cent.		Per cent.		Per cent.
Red	32.4	Green	3.8	Violet	41.5
Yellow	8.8	Blue	25.0		

"The transmission of each color is, of course, 100 per cent., minus the per cent. of absorption for the color under consideration."

The Index of Refraction.

The amount and character of convergence or divergence obtainable in a lens depends upon the extent and nature of its surface-curvatures and the *index of refraction,* which is the ratio between the sine of the angle, i, of incidence and the sine of the angle, r, of refraction for a ray of light passing from one transparent medium into another, Fig. 5, which is the cross-section of a glass plate surrounded by air.

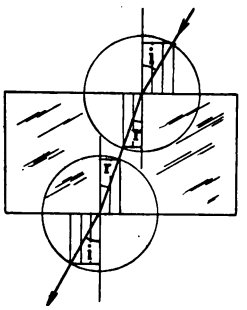

Fig. 5.

It is a *constant,* whose value depends on the nature of the two media separated by the refracting surface, provided the media have the same properties in all directions and the kind of light is specified. This, **Snell's law,** established by Willebrod Snellius, is expressed by the equation,

$$\sin i/\sin r = n, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{I.}$$

the index of refraction, or refractive index of a medium exposed to the incident light having a *specified* wave-length. When the light, before meeting the surface, has been travelling in a vacuum, the constant is termed the refractive index of the medium in which the refraction occurs. When light is incident in air upon a transparent medium, the index of the refraction is practically the same as if the light had been incident in a vacuum upon the same surface. Thus, the above ratio for a ray passing from air into water is about 4/3, or more exactly, 1.336. The index of ordinary crown-glass is approximately $3/2 = 1.5$. The indices more generally used in America are 1.507 and 1.523.

When n is greater than unity, it follows that the angle, r, is less than the angle, i, and the refracted ray makes an angle with the normal which is smaller than the angle of incidence. In fact, the incident rays at the point of impact upon the surface may be said to be bent toward the normal within the medium. This deflection generally

occurs when light passes from a rarer to a denser medium; so that the latter is called the *optically denser medium,* whatever the mechanical densities of the media may be. Conversely, when light is deflected at the surface of separation of two media so as to be bent away from the normal, the index of refraction is less than unity, and the second medium is said to be *optically rarer,* or less dense, than the first.

When light is incident in a vacuum (or in air), on a transparent surface, it is nearly always deflected toward the normal, so that the refractive indices of nearly all transparent media are greater than unity. Light can also penetrate to a small depth into a metal, and in this case it is sometimes bent away from the normal (in the case of sodium, gold and silver), and sometimes toward the normal (in the case of platinum and iron).

Surface-Curvature.

Since the refractive index is a constant for any particular substance, it follows that the deviation incurred by a light-ray is wholly dependent upon the inclination of the refracting surface to the incident light, and in order that the deviation produced by a simple lens may be symmetric with respect to the axis of an incident pencil of light, the surface will require to be correspondingly curved in all of its meridians. Such a surface conveniently lends itself to the mechanical production of a lens, as it may be produced through rotation of an arc of any circle whose center is located upon the axis of revolution, thus producing a spheric surface of any desired radius.

A lens thus formed is called a spheric lens, and is universally used both in spectacles and other optical instruments. However, in a *thin* primitive lens one of the surfaces is either spheric or cylindric, the other surface being a plane, and when the lens thus formed is thickest in the center, it produces convergence and is called a convex, convergent, collective, or plus (+) lens; whereas, if the lens is thinnest in the center, it produces divergence and is called a concave, divergent, dispersive, or minus (—) lens. However, Mr. L. W. Bugbee, of Southbridge, Mass., has shown that the thickness of a lens may be sufficiently increased to reverse these properties.

The terms *plus* and *minus,* when used in this sense, correspond with the Continental Convention; an agreement among physicists of Continental Europe that all distances measured from a given point, such as the center of a thin lens, shall be counted positive when they agree in direction with the propagation of the incident light, and negative, if in the opposite direction. Therefore, convex lenses have positive (+) focal lengths, and concave lenses, negative (—) focal lengths;

this conception being now universal among ophthalmologists and opticians throughout the world. However, in many English works on geometric optics distances are counted positive when measured in a direction opposite to that of the incident light, so that the significance of the signs plus and minus is thereby reversed and tends to unnecessary confusion in the minds of students who may have recourse to differing text-books.

The six diagrams in Fig. 6 represent the principal sections of convex and concave lenses, whose surfaces may be either spheric or cylindric, classified in two groups as being typical in form to produce either convergence or divergence of light.

Typical Lenses.

I. (1) Plano-convex or convexo-plane, each of these terms, applied to the same lens, meaning that the first mentioned surface faces the

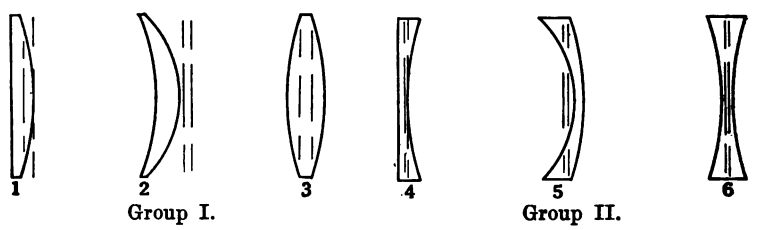

Fig. 6.

incident light. Similarly, (2) concavo-convex or convexo-concave lens, or *meniscus,* which term can properly only be applied to a *convex* lens whose section is a crescent, since it derives its name from the cusps (points) of the crescent moon. (3) Double convex or biconvex lens.

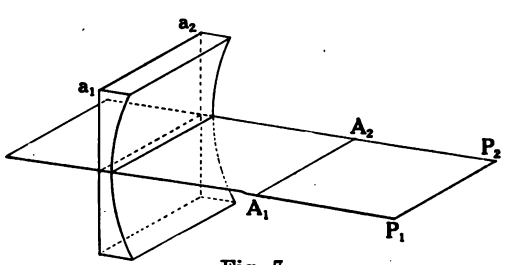

Fig. 7.

II. (4) Plano-concave or concavo-plane, each of these terms used fo the same lens, meaning that the first mentioned surface faces the inci dent light. Similarly, (5) concavo-convex or convexo-concave lens, a *contrameniscus,* which is opposite in form and action to the cresc lens. (6) Double concave or biconcave lens.

Spheric lenses produce either real or virtual *image-points,* whereas cylindric lenses produce similarly characterized *image-lines.*

In a cylindric lens, Fig. 7, the *maximum* refraction is strictly confined to those parallel principal sections, between a_1 and a_2, that are perpendicular to the rotary axis, A_1 A_2, through which the surface-curvature is generated, so that the image-points of contiguous sections establish an array of points to constitute the image-line, P_1 P_2, which is parallel to the cylinder's axis, A_1 A_2. Therefore, the image-line, P_1 P_2, and the axis, A_1 A_2, of the cylindric lens are located in a plane in which refraction does not occur; wherefore, linear dimensions that are located in it remain unchanged when viewed by the normal eye through the lens. For this reason cylindric lenses are used to counteract *simple astigmatism* of the human eye, which is then optically defective only in one plane or principal meridian that may have any angular inclination to the horizon. Hence, it is necessary to specify the position of the cylinder's axis with respect to the horizon, and which is usually counted *clock-wise* from it. As this position is generally read from the graduations on the outside of a trial-frame, they are engraved in the opposite direction.

In the optical trade the term *periscopic,* suggested by Wollaston, was formerly applied to the menisci and contramenisci originally recommended by Hertel, but it is now generally used merely to designate their shallower forms. The form of the lens greatly influences the definition of the image, it having been mathematically determined that the biconvex or biconcave lens producing the least aberration (*Principles of Geometrical Optics,* Southall) has a radius of curvature for the front surface 1/6th of the radius of the back surface. The plano-convex or plano-concave form is nearly as free from aberration; whereas the meniscus and contrameniscus, when given the proper proportions of surface-curvature, produce less aberration for obliquely incident pencils, and therefore give a wider field of distinct vision than lenses of any other form. See section on *Wide-Aperture Lenses.* The concave surfaces of the menisci, or the lesser curvature of biconvex lenses, in case a difference in the surfaces exists, should always be worn in proximity to the eye.

The term "meniscus" is much abused by opticians, through its incorrect application to a *concave* lens having pronounced counteracting curvatures (concavo-convex), and evidently from the desire to specify a more deeply curved periscopic lens, in distinction from the ordinary periscopic lens with lesser proportions of curvature. *Deep meniscus* would be a better term for the *crescentic lens* of accentuated curvature, which is always *convex;* whereas, *deep contrameniscus,*

being counter-disposed in profile and effect to the crescentic lens, would be more correctly applied to a *concave* lens in which the concave and convex surfaces are of very pronounced curvature. Nor should the deep meniscus or deep contrameniscus be called a toric lens, as the latter must necessarily have a toric surface. Such errors give evidence of the need for more profound knowledge among persons engaged in a vocation requiring the proper use of technical terms.

A glass lens having two unequal spheric surfaces is shown in Fig. 8, with its surfaces symmetrically oblique to each other, except at two

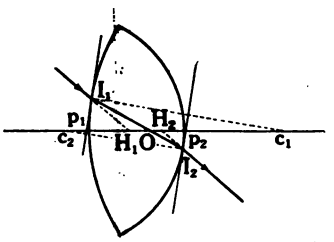

Fig. 8.

opposite points, p_1 p_2, defining the thickness of the lens upon a line, the principal axis, c_1 c_2, or *optic axis,* which also passes through the centers of curvature, c_1 and c_2.

The points, I_1 and I_2, upon the opposite surfaces lie in parallel planes that are perpendicular to the parallel radii, c_1 I_1 and c_2 I_2; hence, the emergent ray at I_2 is also parallel to the incident ray at I_1, and emerges from the lens without change in direction, the same as through a plate. The refracted ray, I_1 I_2, within the lens, cuts the optic axis at a point, 0, called the *optic center,* which is nearer to the surface having the shorter radius, though it is exactly in the middle of the lens when the curvatures are alike.

The prolongations of the incident and emergent rays also cut the axis, c_2 c_1, at two points, H_1 and H_2, called the *principal points.* In fact, all similar lines which are obliquely directed through these points are called *secondary axes,* and connect correlative points in the *object-space* and *image-space* on opposite sides of the convex lens.

Cardinal Points of an Optical System.

There are two distinguished pairs of conjugate axial points, introduced by Gauss, the most important of which are the principal points, and to which Listing has added another pair, the so-called nodal points. The six cardinal points are the two *principal foci,* the two *principal points* and the two *nodal points.* The *first principal focus,*

F_1, is the point on the principal axis where the incident rays intersect, or would intersect if produced, which emerge from the system parallel to the axis, Fig. 9, and in which the final medium has a different refractive index from the first medium, for instance, as in the human eye. See **Refraction of the eye.**

The second *principal focus,* F_2, is the point of axis-intersection of the emergent rays, whose incident direction has been parallel to the principal axis. The *principal points* on the axis are such that, when an incident ray (produced if necessary) passes through the first principal point, H_1, the corresponding emergent ray (produced if necessary) passes through the second principal point, H_2, but the incident and emergent rays are not necessarily parallel to each other. The *nodal points* are two points on the principal axis so disposed that the ray which before refraction is directed toward the first nodal point,

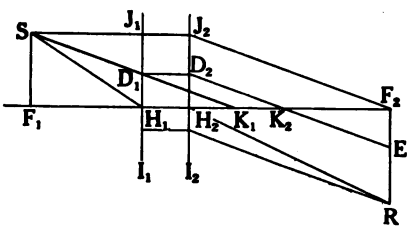

Fig. 9.

K_1, appears, after refraction, to come from the second nodal point, K_2, and follows a direction parallel to its incidence to the system. In each pair of principal points, and in each pair of nodal points, respectively, each point is the image of the other. The distance between the two nodal points is equal to the distance between the two principal points. The two planes drawn through F_1 and F_2, and H_1 and H_2, which are at right angles to the principal axis for axial pencils, are called the two focal planes and the two principal planes, respectively. The rays that originate, or appear to originate, from a point in the *first focal plane,* are, after refraction, parallel to each other and to the lines of direction, K_1 D_1 and K_2 D_2. The incident rays which are parallel to each other, intersect, after refraction, in some point on the *second focal plane.* This point is where the corresponding line of direction cuts the second focal plane. The two *principal planes* are so disposed that the directions of an incident ray and its corresponding emergent ray cut the two principal planes in two points, J_1 and J_2, on the same side and at the same distance from the principal axis. The second principal plane is the optical image of the first principal plane and vice-versa. The principal planes are called *planes of unit magnifica-*

tion, since an area of definite shape and size in the first principal plane produces a virtual image of precisely the same shape and size in the second principal plane. They are the only two conjugate images which have the same size and are situated on the same side of the principal axis. The *first principal focal distance,* F_1 H_1, is the distance between the first principal focus, F_1, and the first principal point, H_1; the *second principal focus distance,* F_2 H_2, being the distance between the second principal focus, F_2, and the second principal point, H_2.

Cardinal Points of a Lens.

Specifically, however, for a thick lens of any form, *in air,* the principal points and nodal points are synonymous, since they are both merged into one point, as, for instance, in the biconvex lens, Fig. 10,

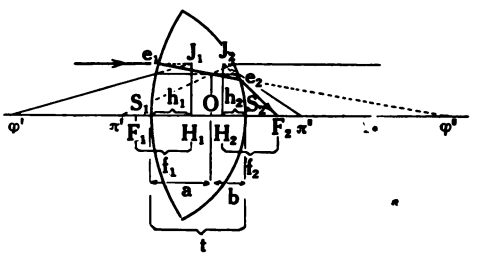

Fig. 10.

wherein the principal planes, H_1 J_1 and H_2 J_2, are conjugate images, of equal size, of the plane erected in the optic center, O. Hence, all rays which before refraction are directed toward a point, J_1, in the first principal plane, seem, after refraction, to come from a point, J_2, in the second principal plane, which is situated on the same side of the axis as the first, and equally distant from the axis.

The second principal focus, F_2, behind the lens, is the point at which are focused all incident parallel rays in front of the lens, since the refracted ray, e_1 e_2, within the lens, is again refracted at e_2 to produce the emergent ray, e_2 F_2, which appears to proceed from the point, J_2, of the second principal plane.

Similarly, parallel rays incident to the lens from the opposite direction have their focus at F_1, which is as far from H_1 as F_2 is from H_2. Consequently, the two principal focal distances, $H_1F_1 = f_1$, and $H_2F_2 = f_2$, of a lens, in air, are equal, irrespective of the surface-curvatures, and are designated simply by f. However, it is to be remembered that this distance, f, on either side of the lens, is always to be counted from its correlative principal point, H_1 or H_2. The

following equations for a thick biconvex lens are obtained through suitable transformation of those given by Landolt (*The Refraction and Accommodation of the Eye.*—E. Landolt, Edinburgh, 1886), and through which it is found that the optic center, O, of the biconvex lens divides its thickness, t, in two parts, a and b, whose values are

$$a = \frac{tr_1}{r_1 + r_2}, \text{ and } b = \frac{tr_2}{r_1 + r_2}.$$

Therefore, when the radii, r_1 and r_2, of the surfaces, S_1 and S_2, are made equal to each other, a is equal to b, and consequently O is midway between the surfaces. When the radii are unequal, the optic center is nearer to the more convex surface.

The distances of the principal points, H_1 and H_2, from the surfaces, S_1 and S_2, are represented by h_1 and h_2, respectively, so that

$$h_1 = \frac{tr_1}{n(r_1 + r_2) - t(n - 1)}, \text{ and } h_2 = \frac{tr_2}{n(r_1 + r_2) - t(n - 1)},$$

these equations being derived from the corresponding general formulæ:

$$h_1 = \frac{tr_1}{n(r_1 - r_2) - t(n - 1)}, \text{ and } h_2 = \frac{tr_2}{n(r_1 - r_2) - t(n - 1)}, \quad \ldots . \text{II.}$$

which apply to a thick lens of any form, when r_1 and r_2 are given their proper signs. Thus, if $r_1 = r_2$, h_1 is also equal to h_2; wherefore, the principal points in an equiconvex lens are equally distant from its surfaces; and, in case the refractive index $n = 1.5$, and t is comparatively small, h_1 and h_2 are equal to one-third of the thickness.

The distances, a and h_1, are to be taken positively to the right and negatively to the left of the first surface, S_1; and b and h_2, positively to the left and negatively to the right of the second surface, S_2. Their dimensions, deduced from the preceding formulæ for a biconvex lens, become positive, so that the optic center and the principal points are inside of the lens. In the case of a meniscus, a and h_1 become negative, and b and h_2, positive, thus placing the optic center and the principal points outside of the lens, near its convex side. As the radii of the meniscus are both positive, the general formula, II., applies without change of signs.

The formula for the principal focal length of a thick biconvex lens whose radii differ is

$$f = \frac{r_1 \; r_2}{(n-1)\,(r_1 + r_2) - \dfrac{t(n-1)^2}{n}},$$

being derived from the general formula for a thick lens:

$$f = \frac{r_1 \; r_2}{(n-1)\,(r_2 - r_1) + \dfrac{t(n-1)^2}{n}}, \quad \ldots\ldots\ldots\ldots \text{III.}$$

after the proper signs for r_1 and r_2 have been introduced.

If, in the latter equation, the thickness, t, is reduced to nil, then

$$f = \frac{r_1 \; r_2}{(n-1)\,(r_2 - r_1)}.$$

Therefore,

$$\frac{1}{f} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{IV.}$$

which is the general formula for the refraction of a very thin lens of any form whose focal length is f.

In the plano-convex and plano-concave lens, the first principal point is at the vertex of the curved surface; whereas, the second principal

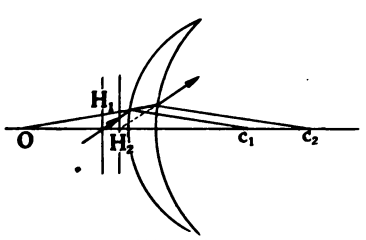

Fig. 11.

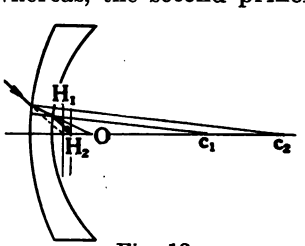

Fig. 12.

point is in the interior of the lens. In the meniscus and contrameniscus the principal points are not in the interior of either lens, but at definite distances on a particular side of it.

Thus, the meniscus, Fig. 11, has its principal points, H_1 and H_2, on the convex side; whereas for the contrameniscus, Fig. 12, they are

on the concave side. In fact, this difference in the positions of the principal points, even in thin lenses of this kind, frequently affects the acuteness of vision when they are substituted as equivalents for thin biconvex or biconcave trial-lenses, respectively. This is due to the fact that their respective principal points, from which the focal lengths are measured, do not occupy the same positions with respect to the plane of the frame supporting either lens before the eye, so that the image of the lens can not in both cases coincide with the ocular far point.

This is more fully explained in the section on *Vertex-Refraction.*

Thin Lenses.

Ophthalmic lenses are usually very thin, so that the thickness may be considered a negligible dimension * with respect to the focal length, which is then counted from the optic center toward either focal point, F_1 or F_2, at the same distance, f, on either side of the lens. Either of these points may, therefore, be the principal focus, namely, the focus of rays that are incident to the lens parallel to its axis. For instance, an incident ray proceeding from the first principal focus (F_1 negative), or toward that point (F_1 positive) is rendered parallel to the axis after refraction through the lens; whereas an incident ray parallel to the axis gives rise to a refracted ray which virtually proceeds from the second principal focus (F_2 negative), or which actually passes through that point (F_2 positive).

As parallel incidence corresponds to the second principal focus, F_2, the latter is universally used to designate the *principal focal length,* f, of a convex or concave lens whose refractive index is n, and whose radii of curvature are r_1 and r_2. Their relation to each other is:

$$\frac{1}{f} = (n-1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

The refraction, $\frac{1}{f}$, also expresses the *power of the lens,* which is proportional to the difference between the powers of the lens-surfaces (see section on the *Dioptry*). The means by which this equation is established, together with other general formulæ given in the text, may be found in any work on geometric optics.

However, great care should be exercised when numeric values are therein substituted for r_1 and r_2; it being necessary to remember that

* Excepting in convex lenses having a focal length shorter than 6 inches.

the optic center of the thin lens is the zero-point of the dimensions of curvature, and which are to be reckoned positive when their direction from the center agrees with the direction of the light's propagation. Of course, a direction opposite to this calls for a negative dimension. This is necessary, in order that convex lenses shall be considered positive, and concave lenses, negative. For instance, when the preceding equation is numerically applied to lenses whose front surface (the one exposed to incident light) has a radius r_1, the signs for the radii, r_1 and r_2, with respect to the optic center, are to be chosen as follows:

Biconvex type, $+ r_1$ and $- r_2$, which gives $+ f$, a positive lens.

Biconcave type, $- r_1$ and $+ r_2$, which gives $- f$, a negative lens.

Plano-convex type, $r_1 = o$, and $- r_2$, which gives $+ f$, a positive lens.

Plano-concave type, $r_1 = o$, and $+ r_2$, which gives $- f$, a negative lens.

Meniscus type, $+ r_1$ and $+ r_2$, as $r_1 < r_2$, which gives $+ f$, a positive lens.

Contrameniscus type, $+ r_1$ and $+ r_2$, as $r_1 > r_2$, which gives $- f$, a negative lens.

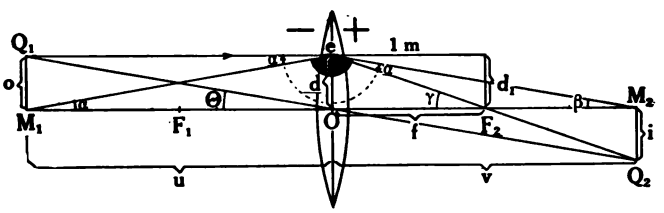

Fig. 13.

The *conjugate foci* of a lens are two points, M_1 and M_2, Fig. 13, so situated that the rays emitted from a luminous body or illuminated object at either point are refracted by the lens to the other. Therefore, the angle at e, shown densely shaded between the rays, M_1e and eM_2, is a *constant,* depending upon the curvature and the refractive index of the lens. For instance, if M_2 moves to F_2 (the second principal focus), M_1 moves in the same direction to infinity, because both sides of the dense angle simultaneously turn clockwise around e through the same angle, α, and thus establish the corresponding angle, γ at F_2. The angles at M_1, M_2 and F_2 are so related that $\alpha + \beta = \gamma$, as shown on the right side of the diagram; so that,

$$\tan \gamma = \tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$$

However, instead of this, *for very small angles,* the tangent may be independently substituted for the arc of each angle; wherefore, $\tan \alpha + \tan \beta = \tan \gamma$, and whose linear dimensions in the diagram give:

$$\frac{d}{u} + \frac{d}{v} = \frac{d}{f},$$

in which d, for a radius of 1 meter, represents the tangent of the angle of the refracted ray's deviation from the axis, or from parallel incidence.

Moreover, in order to insure the angles being small, the decentration of e, represented by d, may be made equal to 1 centimeter,* hence

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}.$$

It will later be shown that the decentration, $d = d_1 = 1$ cm., in the principal focal plane of a lens, is also the measure of its prismatic power at the point of decentration, e.

As the dimensions of u and v are measured in opposite directions from the optic center O, it is necessary to make u negative, in case v and f are to be considered positive, or vice versa; wherefore, every type of lens will have a negative and a positive side with respect to the surrounding space.

As the Continental Convention is commonly applied to ophthalmic lenses, and the direction of u from the optic center is then *opposite* to the direction of the incident light, *u must always be taken negative.*

Consequently the preceding equation becomes

$$\frac{1}{v} - \frac{1}{u} = \frac{1}{f} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{V.}$$

This is *the general formula for the conjugate foci,* wherein u is the object-distance and v is the image-distance from the optic center of the lens whose focal length is f.

While it is true that this formula is applicable to either the English or the Continental Convention, yet, when the latter is applied to *numeric values* of u and f, let it be emphasized that, *the dimension u*

* It is evident that the centimeter-value for d may be applied even to a lens of 5 centimeters focal length, as the tangent at the focus, F_2, is then $1/5 = 0.2$, which corresponds to the angle of 11° 18.6; so that all other inclinations of the paraxial incident and refracted rays to the axis will be considerably less than this.

must be given the minus sign for both concave and convex lenses, and that, *f must be given the minus sign for a concave lens,* in order to comply with the Continental conception of sign for direction.

Thus, for a concave lens: $\frac{1}{v} \quad \frac{1}{-u} = \frac{1}{v} + \frac{1}{u} = \frac{1}{f},$

and for a convex lens: $\frac{1}{v} \quad \frac{1}{-u} = \frac{1}{v} + \frac{1}{u} = \frac{1}{f}.$

When numeric values are introduced in the formula V, for a convex lens it is found that the nearer the object is to the lens, the farther the proportionately increased image recedes from it.

Thus, if u is made infinitely great, v is equal to f, so that the incident rays are parallel, and a real, inverted and infinitely small image is formed at the second principal focus, F_2, of the lens. When u is equal to 2f, v is equal to 2f; in other words, when the object, o, is at twice the focal distance from the center of the lens, Fig. 13, the real

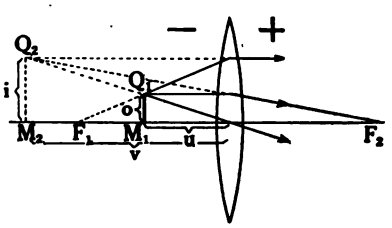

Fig. 14.

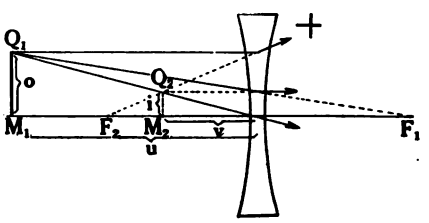

Fig. 15.

and inverted image, i, is also twice the focal distance from it. In case u = f, v is infinitely great, which means that the rays emerge from the lens as a beam of parallel rays directed toward an image that is infinitely large with respect to the object.

However, when the object approaches nearer to the lens than the first principal focus, F_1, say at a distance, u, equal to 2/3 f, then v = − 2f, the minus sign indicating that the image is on the same side of the lens as the object, and, therefore, is virtual and erect, as shown in Fig. 14.

Therefore, it may also be emphasized that, whenever a negative value for v is the result of introducing numeric values for u and f in the formula, V, for a convex lens, it signifies that the image-point, M_2, is on the incident side of the lens; that is to say, it is virtual and on the same side of the lens as M_1, this being the case when the object-point, M_1, is nearer than the first principal focus, F_1.

On the other hand, the formula, V, for a concave lens, will always give a negative value for v, whatever the numeric values of u and f may be, so that the image-point, M_2, is always virtual and on the same side of the lens as M_1. See Fig. 15.

Consequently, every thin lens that increases in thickness towards its periphery has virtual foci; and, vice versa, for the focus of a lens to be real, the lens must be thicker in the middle than at its edge. In other words, a concave lens always produces a virtual image, i, Fig. 15, and a convex lens, a real image, when the object is more remote than the focus, Fig. 13; or a virtual image, in case the object is nearer than the focus, Fig. 14. In fact, the object and the image are inseparably linked together, by means of the conjugate foci, in the so-called object-space and image-space, which, as shown in Figs. 14 and 15, may also occasionally occupy the same territory.

Magnification.

As the sizes of the object, o, and of the image, i, are reciprocal values depending upon the distances u and v, the so-called *lateral magnification,* M_1, is expressed by the equation:

$$M_1 = \frac{i}{o} = \frac{v}{u} = \frac{f}{u+f} = \frac{f-v}{f}.$$

It may have any value, depending upon the position of the axial object-point, and is the measure of the actual magnification. When the semi-diameters of o and i, respectively, are projected to *opposite sides* of the *axis,* the image is *real* and *inverted;* whereas, if these dimensions are on the same side of the axis, the image is *virtual* and *erect.* In the latter case the image is not really formed, but only appears to be formed to the eye that sees it, therefore, the size of the virtual image is only apparent.

However, the apparent size of an object or its image, whether real or virtual, may be determined by the tangent of the visual angle under which it is seen, so that, in case the object is too distant or inaccessible to be measured, the equation above cited may be given the form:

$$M_o = \frac{i}{\tan \Theta} = f \text{ (approximately)}.$$

This is called the *objective magnifying power,* which is the ratio of the linear size of the image, i, Fig. 13, to the *apparent size of the*

object expressed through the tangent of the slope-angle, Θ, of the chief ray that proceeds from the outermost object-point, Q_1. It differs from the so-called *subjective magnifying power,* M_s, which is applied to instruments designed to be used subjectively in conjunction with the eye for the purpose of amplifying vision, as for instance, in the case of the ordinary magnifying glass, or the microscope. In this event, it is the ratio of two individually considered visual angles subtended at the nodal point of the eye, namely, where one of the angles, Θ', corresponds to the image viewed in the instrument, and the other, η, to the object as seen by the naked eye at the distance of distinct vision.

Hence,

$$M_s = \frac{\tan \Theta'}{\tan \eta} = \frac{i}{v}\,\frac{a}{o} = \frac{a}{v}\,\frac{i}{o}.$$

This ratio involves the so-called "distance of distinct vision," a, which, being variable among different individuals, makes it necessary to conventionally adopt 25 centimeters or 10 inches as the standard distance of distinct vision for the normal eye. However, even where allowance is made for the different distances of distinct vision among far-sighted and near-sighted observers, "Abbe has pointed out that both observers, looking through the instrument, will, as a matter of fact, view the image of the same object under the same visual angle; so that whatever difference there may be in the magnifications is to be found, not in the instrument itself, but in the different organs of sight that are employed in conjunction with the apparatus." (*Principles of Geometrical Optics,* Southall.) Therefore, as the angle subtended at the ocular nodal point by the object seen by the naked eye has nothing to do with the instrument, it is convenient to substitute its value, $\tan \eta$, by its equivalent, namely, the linear dimension, o, of the object divided by the distance, a, of distinct vision, or o/a, when the ratio for the subjective magnifying power becomes $\frac{\tan \Theta' a}{o}$, the product of two factors, one of which (a) depends entirely on the eye of the observer. Consequently, Abbe ignores the variable factor, a, having shown that, from a strictly scientific point of view, his equation for the *intrinsic magnifying power,*

$$M_1 = \frac{\tan \Theta'}{o} = \frac{1}{f} \text{ (approximately)},$$

is far superior, since it is strictly a constant of the instrument itself. In fact, in the special case, when the plane of the exit-pupil of the optic system coincides with the secondary focal plane, and the nodal point of the eye is situated at the secondary focal point of the system, Abbe's formula is strictly true for an instrument on the order of the microscope.

Equivalent Lenses.

When two lenses are separated upon a common axis by an appreciable distance, d, a thin single lens can be found, which, when placed at a suitable fixed point, will produce an *image of the same size, but not generally in the same position,* as that produced by the combination. This lens, with a focal length f, is said to be equivalent to the combined lenses whose focal lengths are f_1 and f_2, and is expressed by the equation:

$$\frac{1}{f} = \frac{f_2 + f_1 + d}{f_1 f_2} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{d}{f_1 f_2}, \quad \text{.....VI.}$$

which applies when either f_1 or f_2 is negative, or when both are negative. However, in ophthalmic practice, where lenses of moderate power are superposed in the trial-frame, the distance, d, between them may be neglected, so that their equivalence is expressed by

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} \quad \text{.....VII.}$$

In other words, the equivalent lens, whose focal length is f, when placed in the position occupied by two such lenses in the trial-frame, practically produces the same convergence or divergence from the same point as that produced by the combined lenses.

Moreover, for purposes of rapid calculation, it is convenient to use the reciprocals of the focal lengths rather than the focal lengths themselves. Thus, if D is used to represent the power, 1/f, of the lens whose focal length is f, and D_1 and D_2 are the powers of the lenses whose focal lengths are f_1 and f_2, respectively, the previous equation becomes

$$D = D_1 + D_2 \quad \text{.....VIII.}$$

Therefore, it is also apparent that, if D_1 and D_2 are equal powers of very thin lenses, the one being convex $(+D_1)$ and the other concave $(-D_2)$, the refraction will be nil, or equivalent to a plane glass Consequently, the combined effect of D_1 and D_2 is one of neutralization

However, this principle does not apply to lenses of widely different form or of appreciable thickness, since the power or focal length of a lens depends upon three factors, the radius of curvature, the refractive index and the thickness of the glass; hence the latter must be taken into consideration as soon as it becomes an appreciable percentage of the focal length. This will be more fully explained in the sections on *Vertex-Refraction and Neutralization.*

A pair of lenses is said to be *congeneric* when both lenses are either convex or concave, and *contra-generic* when one lens is convex and the other, concave. Thus, a pair of superposed contra-generic lenses of the same convergent and divergent power, are, within certain limits mentioned in said sections, capable of neutralizing each other; from which it follows that a convex lens, say + 2 D., will counteract the effect of a concave lens, — 2 D.

The Dioptry.

The abbreviation "D." signifies the unit of refractive power, originally in French called dioptrie and since Anglicised to dioptry. It represents the power of a lens whose focal length is one meter = 100 centimeters, or 39.37 U. S. standard inches. As the power of a lens increases, its focal length decreases, and vice versa; or in other words the power of a lens is in inverse proportion to its focal length. Therefore, in the inch-system the power of a lens is *approximately* expressed through the fractions 1/40, 1/20, or 1/10 for a lens whose focal length is 40, 20 or 10 inches; whereas, in the metric system, for a lens whose focal length is 10, 25 or 50 centimeters, the dioptral power is expressed by 100/10 = 10 D., 100/25 = 4 D., or 100/50 = 2 D., respectively. The metric or *dioptral* system is far more convenient since the superposition of low powered lenses so frequently resorted to in ophthalmic practice renders it easier to add their powers in dioptries than to add fractions representing their powers in the inch-system, which, therefore, is now less frequently used. The adjective dioptral refers specifically to lenses and prisms numbered in dioptries and is thus used in distinction to *dioptric,* a term which applies to refraction in general and without reference to any specific unit of measurement. The focal length of any dioptral lens is readily found through dividing 100 cm. or 40 inches by its given dioptral power For instance, 100 cm. divided by 2 D. = 50 cm. focal length; or 40 in

divided by 2 D. = 20 in. focal length. Similarly, 100 cm. divided by 0.25 D. = 400 cm.; or 40 in. divided by 0.25 D. = 160 in. focal length. On the other hand, the power of a lens whose focal length is given, say 50 centimeters or 20 inches, is found through dividing 100 cm. by its given focal length in centimeters, or 40 in. by its given focal length in inches; thus, 100 cm./50 cm. = 2 D., or 40 in./20 in. = 2 D.

It may be repeated that it is misleading to apply the term *"lens"* (which implies a focus) to a plane glass or to one whose curved surfaces are parallel, or the term *"diopter"* to lenses of the dioptral system. "Diopter," is not sufficiently specific, because it may be any one of several instruments used by engineers or surgeons. Therefore, the preference should be given to dioptry, which is certainly unequivocal in its meaning. Moreover, it should be remembered that all lenses, no matter which system of numbering is used, have dioptric (refractive) power, whereas, those of the metric system have specifically *dioptral* power.

As a spheric surface may be thought to be generated through rotation of a circular curve upon its radius, which axis of revolution for a lens is the optic axis, the refraction of a spheric lens is uniformly the same in every *principal section*—a plane in any meridian containing the axis. However, this refraction depends upon the refractive powers contributed by the two opposite surfaces of the lens, and which are often separately considered in the construction of ophthalmic lenses. For instance, in a thin meniscus having a + 3 D. spheric surface on one side, and a — 1.25 D. spheric surface on the other, the total refraction is + 1.75 D. Commercial periscopic lenses are uniformly made with one concave surface, — 1.25 D.; whereas, deep menisci and contramenisci are usually, — 6 D., and, in special instances, — 9 D. The former uniformity facilitates the addition of reading segments from stock for bifocal lenses.

In any event, the power of a thin lens is equal to the algebraic sum of the refractive powers of its opposite surfaces.

Landolt's table, in which the dioptral and the inch systems are compared, is here given.

OPHTHALMIC LENSES AND PRISMS

INCH-SYSTEM AND DIOPTRAL EQUIVALENTS.
Computed by E. Landolt.

OLD SYSTEM.				NEW SYSTEM.			
I. Number of the Lens, Old System.	II. f = Focal Distance in Engl. in. for n = 1.53.	III. f = Focal Distance in Millimeters.	IV. D. Equivalent in Dioptries.	V. Number of the Lens, New System.	VI. f = Focal Distance in Millimeters.	VII. f = Focal Distance in Engl. inches.	VIII. Corresponding Number of the Old System.
72	67.9	1724	0.58	0.25	4000	157.48	166.94
60	56.6	1437	0.695	0.5	2000	78.74	83.46
48	45.3	1150	0.87	0.75	1333	52.5	55.63
42	39.6	1005	0.99	1	1000	39.37	41.73
36	34	863	1.16	1.25	800	31.5	33.39
30	28.3	718	1.39	1.5	666	26.22	27.79
24	22.6	574	1.74	1.75	571	22.48	23.83
20	18.8	477	2.09	2	500	19.69	20.87
18	17	431	2.31	2.25	444	17.48	18.53
16	15	381	2.6	2.5	400	15.75	16.69
15	14.1	358	2.79	3	333	13.17	13.9
14	13.2	335	2.98	3.5	286	11.26	11.94
13	12.2	312	3.20	4	250	9.84	10.43
12	11.3	287	3.48	4.5	222	8.74	9.26
11	10.3	261	3.82	5	200	7.87	8.35
10	9.4	239	4.18	5.5	182	7.16	7.6
9	8.5	216	4.63	6	166	6.54	6.93
8	7.5	190	5.25	7	143	5.63	5.97
7	6.6	167	5.96	8	125	4.92	5.22
6½	6.13	155	6.42	9	111	4.37	4.63
6	5.6	142	7	10	100	3.94	4.17
5½	5.2	132	7.57	11	91	3.58	3.8
5	4.7	119	8.4	12	83	3.27	3.46
4½	4.2	106	9.4	13	77	3.03	3.21
4	3.8	96	10.4	14	71	2.8	2.96
3½	3.3	84	11.9	15	67	2.64	2.8
3¼	3.1	79	12.7	16	62	2.44	2.59
3	2.8	71	14	17	59	2.32	2.46
2¾	2.6	66	15.1	18	55	2.17	2.29
2½	2.36	60	16.7	20	50	1.97	2.09
2¼	2.1	53	18.7				
2	1.88	48	20.94				

Vertex-Refraction.

The need for a fixed point of orientation from which to measure the focal length of a lens of any form placed before the eye was first recognized by Dr. Moritz von Rohr, physicist, of the firm of Carl Zeiss, Jena, Germany, who suggested that it shall be the pole or vertex of the lens-surface facing the cornea; this being in precise agreement with the measurement of prisms suggested by the writer in 1890. In other words, the reciprocal of the distance between the vertex of this surface and the true focus of the lens shall represent its power or so-called vertex-refraction, designated by the symbol D_v. In fact, through such means only is it possible to accurately substitute a deeply curved lens for one of plano or equibispheric form, the increasing popularity of the former having encouraged Dr. von Rohr's investigations with this purpose in view. To the Bausch & Lomb Optical Co. is due the credit for having introduced the vertex system in America.

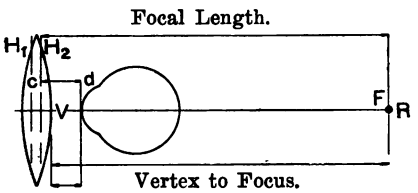

Fig. 16.

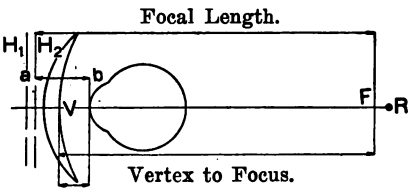

Fig. 17.

In the accompanying figures, 16 and 17, a biconvex lens and a meniscus of equal focal lengths are placed at the same interpolar distance from the eye, but, owing to the second principal plane, H_2, of the meniscus being farther (ab) from the cornea than that (cd) of the biconvex lens, the focal point, F, of the meniscus is correspondingly nearer than the *punctum remotum,* R, of the eye, which demonstrates that the meniscus will require to have a greater focal length, if its focus is to coincide with the ocular far point. Otherwise, as shown in the diagrams, the meniscus of the same focal length overcorrects the hyperopia.

This clearly shows that lenses of the same refractive power are not interchangeable as ophthalmic corrections when they are of materially different forms. In fact, this is especially true when the power exceeds 3.5 dioptries.

Since the positions of the principal points, with respect to the lens itself, are dependent upon both the form and the thickness of the lens, it follows that its focal length is dependent upon two variables un-

known to and neglected by the practitioner who measures ocular refraction.

Up to within a comparatively short time the refractive power of the lens and its distance from the eye have constituted the only data thought to be of moment in such practice, and which, indeed, is still true, provided the focal length of the lens is measured from the vertex of the lens proximate to the eye, instead of from the ever shifting principal points. See end of section on *Cardinal Points of a Lens.* In other words, when the vertex-refraction is substituted for the refractive power, the form and thickness may still be practically ignored. so long as means are provided to measure the vertex-refraction and the distance between the pole of the lens and the cornea.

By vertex-refraction is meant the reciprocal of the distance between the vertex of the lens and its focus, this distance for an equibiconvex (or biconcave) lens, being about ⅓ of the lens-thickness shorter (or longer) than the true focal length; that is to say, the vertex-refraction is correspondingly greater for the convex lens, Fig. 16, than its refractive power. But, as the thickness of ophthalmic lenses below 7D. is a negligible dimension, there is practically not any difference between the refractive power and the vertex-refraction, so that the latter's practical utility is really confined to biconvex lenses of greater power than 7 dioptries and to the thin menisci and contramenisci above 3.5 dioptries.

It is only above these powers, respectively, that the displaced principal points produce a material difference between vertex-refraction and refractive power.

The following tabulation, determined through the subscribed formulæ, sufficiently shows to what extent a difference in the thickness of bispheric lenses affects these values.

COMPARATIVE VALUES OF REFRACTING POWER AND VERTEX-REFRACTION IN STRONG LENSES.

	Ref'g Power in Dioptries.	Vertex-Ref'n in Dioptries.		Neg'ble Differences in Vertex-Ref'n.
1. Equiconcave Lens, 10 D. surfaces, 0.6 mm. thick.....	— 20.04	— 19.96	No thickness, — 20 D.	0.04 diop.
2. Equiconvex Lens, 10 D. surfaces, 7.8 mm. thick.....	+ 19.48	+ 20.55	Don't neutralize — 20 D.	
3. Biconvex Lens, unequal surfaces, 7.8 mm. thick.....	+ 18.98	+ 19.96	from 9.47 D. surface	0.04 diop.
whose surfaces are indicated in column 3.....	+ 18.98	+ 20.02	from 10 D. surface	0.02 diop.

The surfaces of the lens, 3, have been thus chosen in order that its refractive power, + 18.98, shall neutralize with — 20.04 when the 10 D. surfaces of both lenses are in contact. The vertex-refraction of the lenses, 1 and 3, whose refractive powers neutralize with 20.04, are both practically 20 dioptries in vertex-refraction, since hundredths of a dioptry in the last column are negligible values.

Here it is shown that increased thickness of the convex lenses, while reducing the refractive powers, correspondingly increases their vertex-refractions. The reverse is true of the concave lens. Incidentally, when a meniscus exceeds 4.25 D. it is thicker than a biconvex lens of the same power, but if it is a Punktal lens it is proportionately even still thicker among all powers. This lens is described in the section on *Wide-Aperture Lenses.*

When the thickness of a lens exceeds 0.001 of the focal length, it is no longer permissible to compute its power by adding the refractive powers of its surfaces, so that it becomes necessary to incorporate its thickness in the formula:

$$D = D_1 + D_2 - D_1 D_2 \frac{t}{n}, \quad \text{IX.}$$

in which D is the dioptral power of the lens, D_1 and D_2, the powers of its surfaces, t, the thickness, and n, the refractive index of the glass.

The vertex-refraction (D_v) is obtained through the equation:

$$D_v = D. \frac{1}{1 - D_1 \frac{t}{n}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots X.$$

when the surface corresponding to D_1 is remote from the eye. In cas the other side of the lens is to occupy this position, the numeric valu of D_2 is therein to be substituted for D_1.

Through these formulæ it may be easily shown that the thicknes and the distance between two or more superposed lenses have con siderable influence upon the refractive power and the vertex-refrac tion, so that it is also inaccurate to ignore the distance between lense in the trial-frame as is commonly done, especially if the lenses are o appreciable power. In fact, it is absolutely necessary that the pre scribed ophthalmic lens shall have the same vertex-refraction as th lenses combined in the trial-frame. As the disclosures which may b made through these formulæ are of special interest, the first one is her

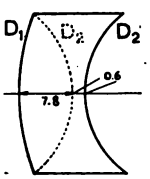

Fig. 18.

Focal length, convex lens.

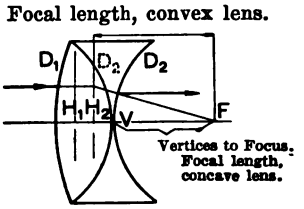

Fig. 19.

applied to determine the refraction of a spherically curved mediun Fig. 18, 8.4 mm. thick, having a concave surface, D_2, correspondin to 10 dioptries of refraction. The refractive index of the glass is 1.50 Now, let it be desired to know what curvature shall be given to the su face, D_1, in order that the refractive power of this medium shall b *nil.* In this instance,

$$D = 0,\ D_2 = -10, \text{ and } \frac{t}{n} = \frac{0.0084}{1.507} = 0.005573 \text{ meters,}$$

$$\text{wherefore, } 0 = D_1 - 10 + D_1 \times 0.05573.$$

$$D_1 = 9.472,$$

the refraction of the first surface of the medium, and which, bein less than $D_2 = 10$, demonstrates that the surfaces are not parallel, bu actually bound a *contrameniscus without power,* its thickness bein

8.4 mm. In order to make this apparent, the curvatures of D_1 and D_2 have been drawn disproportionate to each other. Furthermore, it is quite evident that this absence of refractive power will not be affected if, for convenience of further illustration, a spheric surface of 10 dioptries, corresponding to the dotted line in Fig. 18, is interpolated so as to virtually constitute two contiguous lenses of the thickness, 7.8 mm. and 0.6 mm., respectively. The power, D, of the convex lens thus constituted, when freed in the air, is therefore,

$$D = D_1 + D_2 - D_1 D_2 \frac{t}{n}, \text{ in which case } \frac{t}{n} = \frac{0.0078}{1.507} = 0.0051758.$$

$$D = 9.472 + 10 - 94.72 \times 0.0051758 = 19.472 - 0.4902.$$

$$\therefore D = 18.982,$$

which is the refractive power of the convex lens. The power of the concave lens, 0.6 mm. thick, having 10-dioptry surfaces is similarly obtained through giving the negative sign to the surfaces, D_1 and D_2, so that $-D = -20.04$ is the power which neutralizes $+18.982$, the power of the contiguous convex lens.

As the surfaces of this convex lens are of different powers, it is necessary, in determining its vertex-refraction, to apply the formula, X to each side of the lens, since it will depend upon which side of the lens is to be proximate to the eye. Hence, if the weaker side, 9.472, is placed near the eye, and which would be the case with a lens of this form, the stronger side, 10, is to be substituted for D_1 in the formula:

$$D_v = D \frac{1}{1 - D_1 \frac{t}{n}}, \text{ and since } \frac{t}{n} = 0.0051758,$$

$$D_v = 18.982 \frac{1}{1 - 10 \times 0.0051758} = \frac{18.982}{0.94824}.$$

$$\therefore D_v = 20.02, \text{ or practically } 20\ D_v.$$

Similarly, if the stronger side of the convex lens is proximate to the eye, then 9.472 is the front surface, D_1, so that the vertex-refraction is found to be

$$D_v = \frac{18.982}{0.9509}$$

$$\therefore D_v = 19.962,$$

therefore, also practically 20 D_v., and which is the highest lens-power used in ophthalmic practice.

This conclusively proves that:

Two superposed contra-generic lenses of minimum thickness, having two surfaces in absolute contact and whose dioptral powers effectually neutralize each other, have the same vertex-refraction, even though their refractive powers are actually and unavoidably different.

Conversely, two contra-generic lenses of least thickness and of the same vertex-refraction, whose surfaces are brought into absolute contact, neutralize each other's unequal refractive powers.

Surprising as this may prove to be, it also demonstrates that the vertex-system does not defeat the advantages of the principle advocated by the author and adopted twenty years ago by all but one of the leading lens manufacturers in America, namely, that the stronger biconvex lenses shall be made weaker than standard biconcave lenses of least polar thickness, in order to extend neutralization to all accurately contiguous bispheric lenses, and whose vertex-refractions, for this very reason, are now found to be the same as the dioptral numbers they conventionally bear, and which may be readily verified through measurement by the vertex-dioptrometer made by the Bausch & Lomb Optical Co. Of course, it is to be understood that, among convex lenses of high power their surfaces shall so differ that the stronger side may be brought into effectual contact with the standard concave lens of minimum thickness.

Quite apart from this demonstration, in which provision is actually made for a concave lens-thickness of 0.6 mm., it is self-evident that, if the concave lens, Fig. 19, is made infinitely thin and the closely fitted contiguous lenses of different powers neutralize each other, their focal points must be in the same place, while their vertices likewise coincide. Therefore, the distances between the vertices and the focus are practically equal, so that the vertex-refraction is the same for each lens. It is, of course, to be understood that the aforesaid law applies only to ophthalmic lenses whose surfaces are in close contact and whose apertures are comparatively small. American lens manufacturers have in the past assumed quite different attitudes with respect to the numbering of their biconvex lenses ranging between 7 and 20 dioptries; one optical company having ignored in them the admissibility of neu-

tralization, the other companies having jointly conceded and attempted to provide for it, so that the actual powers of such biconvex lenses have not been uniformly the same for all lenses bearing the same number. This discrepancy has for years been a source of misconception and inconvenience to most opticians, but, fortunately, will no longer exist after the vertex system becomes the standard of comparison to be verified by the vertex dioptrometer.

For instance, it is well known that a + 20 D. biconvex lens of appreciable thickness can not be neutralized by — 20 D., even though each lens actually measures 20 D. upon the optical bench, or an equivalent instrument. See lens, 2, in the preceding tabulation. On the other hand, if the principle of neutralization, for a refractive index of 1.507, is made to apply to a bispheric lens of the therein indicated curvatures and power, a standard biconcave lens (0.6 mm. thick) of — 20.04 D. will be neutralized by a + 18.98 D. biconvex lens (7.8 mm. thick), which, being conventionally numbered + 20 D., will now by the instrument actually measure + 19.96 D_v. and, therefore, practically + 20 D_v. Hence, the + 18.98 D. lens, which neutralizes the — 20.04 D. lens also has, within the negligible error of 4/100ths of a dioptry, the same vertex-refraction, 20 D_v. In short, the dioptral values of all lenses made by manufacturers who originally adopted the author's suggestion, within the limitations herein set forth, are at present actually the equivalents of vertex-refraction.

In fact, the trial-sets made by one of the American manufacturing companies are claimed for this reason to have been numbered, during the past twenty years, to signify either dioptries or vertex-refraction, though the latter was, of course, not to have been anticipated, since Dr. von Rohr has only recently and probably quite unconsciously proven the wisdom of the author's original suggestion. Let it be well understood that there is not any difference between dioptries and vertex-refraction when they are applied exclusively to biconvex and biconcave lenses of really *negligible thickness, or to convex lenses, regardless of power, that are effectually neutralized through actual contact with concave lenses of least thickness,* and only in case they do not thus neutralize is there any discoverable difference in these values among *biconvex lenses* stronger than 7 D. Therefore, in any case where effectual neutralization is in doubt, the vertex-refraction is to be determined by means of the vertex-dioptrometer. In fact, this instrument may be relied upon to secure absolute precision with respect to the measurement and duplication of any lens or combination of lenses in the trial-frame, in terms of vertex-refraction. Moreover, this system of measurement supplies a long felt need in making it

possible to more accurately substitute the menisci and contramenisci for lenses of the flatter types or their combinations, besides obviating recourse to neutralization among deeply curved lenses, and which never did apply to them with any reliable degree of accuracy.

Should these lenses eventually take the place of all other ophthalmic lenses, excepting those in the trial-set, then certainly will vertex-refraction, the vertex-dioptrometer and the instrument for measuring the distance between the poles of the cornea and trial-lens constitute absolutely indispensable accessories to ophthalmic practice. However, in the interim the progressive practitioner will recognize the desirability of having his trial-lenses rated in vertex-refraction, provided they do not accurately neutralize each other throughout the entire range of their powers, and of using a trial-frame in which the distance between the cells is reduced to a minimum, so as to secure the true vertex-refraction of superposed spheric and cylindric lenses when they are to be substituted by menisci or contramenisci. In order to avoid the influence of thickness it is now proposed to materially reduce the diameter of our trial-lenses, which are to be numbered in vertex-refraction and made flat on one side so as to minimize the extent of their separation in the trial-frame. With this change, the principle of neutralization will, of course, all the more readily apply among all powers of paired contra-generic trial-lenses whose curved surfaces may be effectually brought into contact.

The facts disclosed in this discussion justify the conclusion that, from a strictly practical point of view the vertex-system is a reliable monitor to insure the righteous performance of convex lenses, especially those which are stronger than 7 dioptries, and a subtle guide to the correct substitution of a deeply curved lens, either spheric or toric, for any lens or combination of lenses that may be mounted in the trial-frame at a definite distance from the eye.

While it is true that the vertex-system of measurement and the method of neutralization are equally inefficient with respect to disclosing the *true refracting power* of a strong convex lens, yet the present practice of neutralization may be perpetuated as an eminently convenient means of verifying the fact that:

When two bispheric contra-generic lenses in close *surface-contact*, of any power and of minimum thickness, effectually neutralize each other, they are of the same vertex-refraction.

Neutralization.

The thickness of a lens whose power is not restricted can only be considered a negligible dimension in a *standard concave lens* of min-

imum thickness, since all concave lenses between 0.25 D. and 20 D. can be made of the same infinite thinness in the center. In convex lenses, however, there is an unavoidable increase in thickness, which becomes of sufficient magnitude in lenses above 7 D. to conflict with the assumption of negligible thickness. This was explained in the preceding section, though it will here be discussed from a somewhat different view-point.

When the element of thickness is considered, we have the formula for bispheric lenses of equal curvatures:

$$r = f(n-1) + \sqrt{\frac{(nf - t)\, f(n-1)^2,}{n}}, \quad \ldots\ldots\ldots \text{XI.}$$

wherein r is the radius of curvature, n, the index of refraction, f, the focal length, and t, the thickness, in contra-distinction to the formula for neglected thickness, wherein $r = 2f(n-1)$ ..... XII.

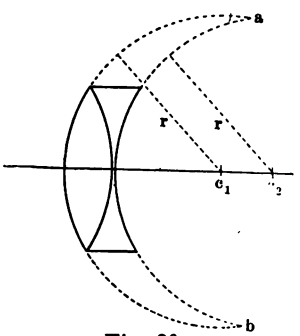

Fig. 20.

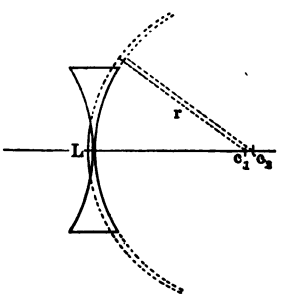

Fig. 21.

Therefore, it is evident that the radius of curvature will be a different one for equibiconvex lenses, in which thickness is considered, from that of biconcave lenses, of the same power, having no appreciable thickness.

The accompanying diagram, Fig. 20, representing a biconvex lens in contact with a biconcave lens of *identical* curvatures, clearly shows that they can not optically neutralize each other, as they really constitute only the central portion of a much larger periscopic *convex* lens or meniscus, and which our imagination can construct upon the dotted lines that are continued to their intersections at a and b.

The diagram also shows that the power of the imaginary meniscus must *increase* with an increase in the thickness of the biconvex lens,

because the anterior and posterior surfaces of the meniscus will be rendered more oblique to each other as their respective centers of curvature, c_1 and c_2, are separated to provide for an increased thickness. The nearest approach to neutralization will, therefore, be secured when the centers of curvature, c_1 and c_2, are as close together as possible, thus making the biconvex lens, L, exceedingly small and thin, as shown in Fig. 21.

However, the lenses in our trial-cases are too large to secure even this *approximate* neutralization. Their diameter of necessity determines the thickness, which must increase with the power. For instance, in a 20 D. equibiconvex trial-case lens of 3.5 centimeters diameter we find the minimum thickness to be 0.75 centimeters. If, therefore, in formula XI, we place, $t = 0.75$, $n = 1.5$, and $f = 5$

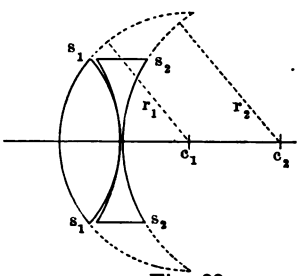

Fig. 22.

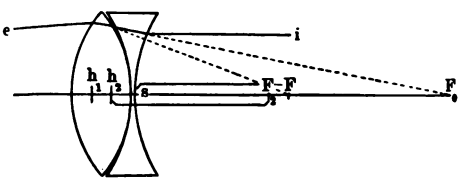

Fig. 23.

centimeters, we obtain 4.87 centimeters as the value of r; whereas, for a 20 D. concave lens, according to formula XII, we find $r = 5$ centimeters.

As the radius is shorter for the convex lens than for the concave lens of the same power, it is evident that their surfaces will actually touch each other only in the center, as exaggeratedly shown in Fig. 22.

Besides, the outer surface, s_1s_1, of the convex lens will be even more oblique relatively to the outer one, s_2s_2, of the concave lens, so that these lenses actually form the center of a stronger convex meniscus than is shown in Fig. 20. Or, viewing it in another light: Parallel rays, i, in Fig. 23, that are incident to the concave lens, are refracted by it as if emanating from the virtual focal point, F, of the concave lens, which is outside of the focal point, F_2, of the convex lens.
Such rays, e, will therefore be rendered convergent, instead of parallel, in passing out of the convex lens, showing that neutralization does not exist.

The focal distance, sF, of the concave lens, one therefore capable of being made infinitely thin, is counted from a single point, s, on

the optic axis in the center of the lens; whereas, in convex lenses of appreciable thickness, it is counted from the *posterior principal point,* h_2, within the lens. In a biconvex lens of equal curvatures, made of glass with an index of refraction $n = 1.5$, it has been demonstrated that the principal points, h_1 and h_2, are separated by a distance equal to one-third of the lens-thickness. (Müller-Pouillet's *Lehrbuch der Physik,* page 160, Braunschweig, 1894.) Therefore, it is obvious that the focal distance of the convex lens will have to be increased by at least one-third of the lens-thickness, so as to have F_2 and F *coincide* for the purpose of effecting neutralization.

With a minimum thickness equal to 0.75 cm., we must consequently add 0.25 to the focal distance, 5, making 5.25 cm. the focal distance of the convex lens. This corresponds to a refraction of 19.047 dioptries. Consequently, *a 19.047 equibiconvex lens of 0.75 cm. thickness neutralizes a 20 D. concave lens of infinite thinness.* To be more accurate, we should actually allow for 0.06 cm. thickness of the concave lens. This would result in the convex lens being even somewhat weaker than 19.047 dioptries.

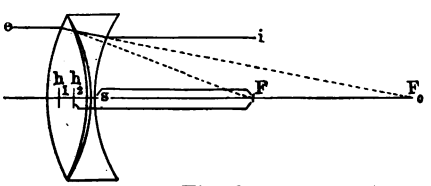

Fig. 24.

However, according to formula XI, a 19.047 D. equibiconvex lens of 0.75 cm. thickness should have a radius $r = 5.121$ cm., so that the superposed neutralizing lenses would actually touch each other at their edges, instead of at the center, as exaggeratedly shown in Fig. 24.

Furthermore, any additional increase in the thickness of the convex lens will be associated with an increase in the distance, h_2F, and will, therefore, call for a corresponding decrease in the power of the convex lens to produce neutralization.

Thus, the tendency will invariably be to over-estimate the power of the convex lens, when an effort is made to determine its power by that of the concave lens which neutralizes it.

Calculation shows that the discrepancies in neutralization of lenses of this kind, varying between 0.25 D. and 1 D., exist in the entire series of convex lenses between 7 D. and 20 D. Consequently, it also follows that the indiscriminate superposition of lenses, so frequently

practised during the subjective method of examination of ocular refraction, is not permissible with lenses of high power.

In other words, lenses of high power are no more capable of being algebraically combined than prisms of high power; a supposed fault for which the practicability of the prism-dioptry was so severely criticized without heed of the injunction that, the desire to multiply any *unit in optics* should be curbed by a knowledge of the fact that all the fundamental optical laws are based upon the assumption and acceptance of *values of limited magnitude,* and that there is, therefore, apt to be a point where *unreasonable* multiplication of an optical unit will contradict the actually existing optical phenomenon.

The general, though erroneous, belief that the entire series of corresponding contra-generic lenses of equal power should neutralize each other has gained such general credence that lens manufacturers have allowed themselves to be swayed by this popular opinion, and

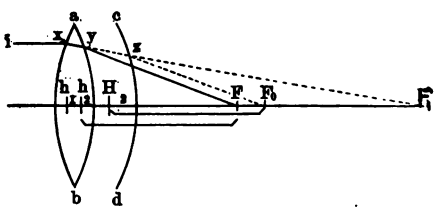

Fig. 25.

as a result have justifiably adopted the principle of making the strong convex lenses weaker than the standard concave lenses, so as to meet the demand for neutralization, and which, under stipulated conditions mentioned later, conforms to the vertex-system of measurement. As has been shown, a 20 D. equibiconvex lens should have a shorter radius than a concave one of the *same* power, yet examination of many of the older trial-cases will reveal the fact that such is not the case, when the surfaces are measured by a suitable gauge. We can, however, gain no reliable information regarding the power of a *strong* convex lens through measurement of its surfaces by means of the spherometer or modern lens-measure, since two lenses of the same curvature, but of different thickness, will not be of the same power. As the thickness increases, the power will diminish for one and the same curvature. This is shown in Fig. 25.

The so-called lens-measure is indeed practicable *only for concave* lenses, since they alone are of negligible thickness, but for thick convex lenses above 7 dioptries this instrument is absolutely unreliable; nor can it be used on any other lenses than those having a refractive index of 1.5.

The incident ray, i, Fig. 25, is refracted by the anterior surface, ab, of the lens in the direction xF_1, and by the posterior surface at y to the focus, F. If the thickness be increased, so as to place the posterior surface at cd, then xF_1 will be refracted by the posterior surface at z to F_0, parallel to yF, since the surface, cd, is of the same radius as ab.

The focal distance, H_2F_0, of the thicker lens, is not appreciably different from h_2F of the thinner lens; in fact, the difference between these distances scarcely amounts to 0.05 cm., for a curvature of 5 cm., in lenses of 0.75 cm. or 1 cm. thickness. Nevertheless, such a difference would be appreciated by the eye in neutralizing the lenses.

This investigation, originally made in 1895, led the author to suggest that all biconvex lenses shall be made weak enough to neutralize with corresponding concave lenses of standard minimum thickness, even though this caused the weaker convex lenses to be numbered higher than their actual dioptral powers, which are, of course, the reciprocals of the distances between the principal points of the lenses and their focal points. However, with the recent introduction of the vertex-system, which substitutes the vertex of the lens for its principal point, the actual dioptral power of the convex lens is equally weaker and similarly ignored, with the fortunate result that the author's suggested weaker convex neutralizing lens is the one whose power is expressed in vertex-dioptries, provided that the biconvex lenses among the higher powers have such unequal curvatures of their surfaces as to permit one of the latter to be brought into close contact with the corresponding surface of the neutralizing equibiconcave lens. In fact, this principle is claimed to have been applied by one of the optical companies in this country. See section on *Vertex-Refraction.*

When convex lenses stronger than 7 dioptries are prescribed, and they are measured by neutralization with concave lenses, we should simply remember that the actual dioptral powers of the stronger convex lenses are always weaker than the dioptral powers indicated by the numbers upon the concave lenses.

On the other hand, if the numbers are intended by the makers to indicate that corresponding contra-generic lenses in the trial-case neutralize each other by actual contact of, at least one pair of, their surfaces, then the numbers also signify vertex refraction.

Prisms.

As the convergence or divergence of light produced by a lens necessarily involves variable degrees of deviation, we shall first consider the simple deviation produced by a prism, which is the inheritent

deviating element at each point of curvature of any lens. In fac every principal section of a lens may be considered to be composed (prisms whose angles vary at successive points of curvature of i surface, or in other words, every meridian of a lens, between i center and its edge, consists of a pyramidal aggregation of prism

Fig. 26.

Fig. 27.

Fig. 26 and Fig. 27, which vary in angle, and consequently in devia ing power, at each succeeding point upon the line of surface-curvatur and at which point a single ray of light is deviated from its incide direction. This deviation or refraction occurs whenever a ray light is obliquely incident to a transparent medium, or passes throu one whose opposite sides embrace an angle.

Therefore, the prism is the simplest form that can be given to medium which will positively insure the necessary oblique inciden

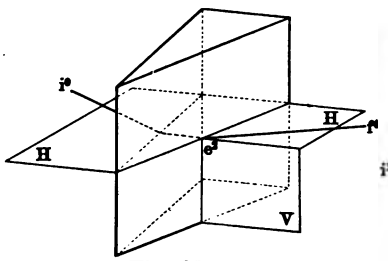

Fig. 28.

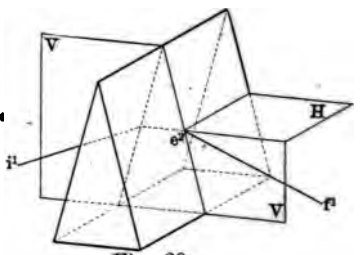

Fig. 29.

to produce angular deviation of a refracted ray of light. In fa any refracting medium included between two plane surfaces tl embrace an angle between them is called a prism, and the line wh the two surfaces meet, or would meet, if produced to form the refra ing angle or *apex-angle,* is called the edge. The base is the s opposite to this apical angle. The refracted ray always lies in same plane with the incident ray. Hence, the incident ray, i, a refracted ray, f, are shown in the horizontal plane, H, Fig. 28, a in the vertical plane, V, Fig. 29; each of these planes being coincid with the principal section in which the greatest refraction is produc

All principal sections of the prism are equal in every respect and parallel, and a line connecting the base with the apex in a principal section is called the *base-apex line*.

These figures illustrate that the refraction is strictly confined to the plane whose intersection with the medium defines the obliquity of

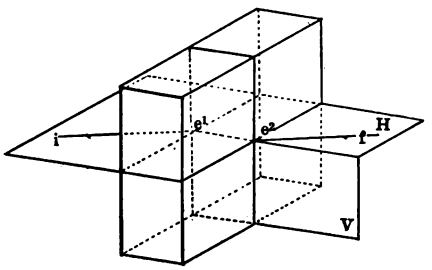

Fig. 30.

its surfaces; whereas, in the right-angled coordinate plane the refraction is passive or nil, because the direction of the ray remains unchanged in passing through opposite parallel surfaces, even though an obliquely incident ray, i, is deviated within a plate, Fig. 30, and from whence it again emerges as a refracted ray, f, parallel to the incident ray.

It is evident that the refraction is active in one and passive in the other plane for a medium whose surfaces are oblique in but *one* plane,

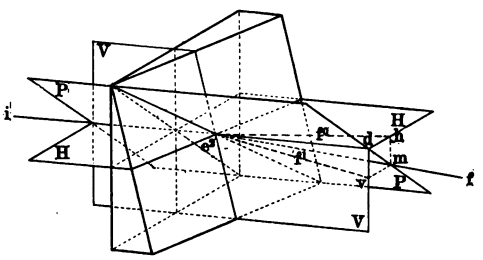

Fig. 31.

so that, to obtain the refraction active in both fixed planes, an obliquity of the surfaces relative to each plane is necessary. In such a medium, Fig. 31, if the refraction is considered merely with regard to the horizontal obliquity of the surface, the refracted ray will assume the direction e_2h, and, if it is considered independently for the vertical obliquity, the refracted ray will assume the direction e_2v. Therefore, with due regard to the obliquity in both planes, the refraction must include both properties of deflection and result in a

refracted ray, f, which is directed to a point, m, defined by projection of the apportioned horizontal and vertical displacements, dh and dv. As this is a prism whose base is really set diagonally to the fixed right-angled coordinate system, the ray, f, must naturally be refracted in the direction of the greatest distance apart of the surfaces, through the point, m, within the diagonally bisecting or oblique plane, P. Consequently, the deviating power of the prism is proportional to the line, dm, which is the hypothenuse of the triangle formed by the forces, dh and dv; hence the equation:

$$dm = \sqrt{(dh)^2 + (dv)^2}.$$

When light passes through a prism that is more dense than air, its emergent pencil at e_2, Fig. 32, is always deviated from the direction of incidence at e_1, toward the base of the prism, while the deviation, δ, also increases with the apical angle. In other words, the axis of the emergent pencil will either really (as in diagrams a and b) or virtually (as in c and d) intersect the plane of the base. In fact,

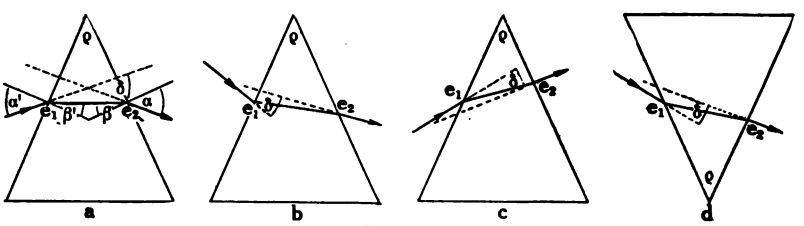

Fig. 32.

the direction of the final deviation, with respect to the base of a prism of any material, depends upon whether the refractive index is greater or less than unity. Thus, for instance, in a very thin prism of gold, whose refractive index is 0.58, the deviation is away from the plane of the base. Therefore, with respect to the transmission of light, gold is less dense than air. The total deviation, δ, is equal to the sum of the angles of incidence, α', and emergence, α, less the refracting angle, ς, of the prism. Hence, in Fig. 32 (a), $\delta = \alpha' + \alpha - \varsigma$; also $\varsigma = \beta' + \beta$, $\sin \alpha' = n \sin \beta'$ and $\sin \alpha = n \sin \beta$, the two latter equations expressing Snell's law of refraction through the refractive index, n. The relation existing between the angle of incidence, α', the angle of emergence, α, the refracting angle, ς, and the refractive index, n, of a prism is expressed by the equation:

$$\sin \alpha = \sin \varsigma \sqrt{n - \sin^2 \alpha'} - \cos \varsigma \sin \alpha',$$

so that the value of any of these factors may be determined when at least three of them are known.

Therefore, the produced deviation, δ, is dependent upon three factors, namely, the angle of incidence, the index of refraction and the angle of the prism, and, as both the index and the angle are constants for any given prism, the deviation may be rendered variable through merely increasing or decreasing the angle of incidence. Consequently, there is a position of the prism, with respect to the incident ray, which will produce the *least* deviation, the so-called *minimum deviation.* In order that a ray shall suffer minimum deviation, the angle of incidence, α', and the angle of emergence, α, must be equal; wherefore, the ray within the prism is also equally inclined to the two faces, Fig. 33; that is to say, the prism must be symmetrically placed with respect to both the incident and the emergent ray.

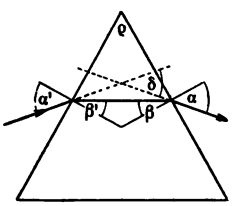

Fig. 33.

When the refracting angle, ς, of a prism is known, and the minimum deviation, δ, has been determined by means of a spectrometer, the refractive index, n, of the substance composing the prism may be calculated by means of the equation:

$$n = \frac{\sin 1/2\ (\delta + \varsigma)}{\sin \varsigma/2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{XIII.}$$

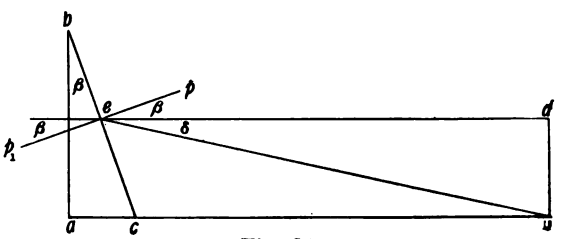

Fig. 34.

In case the incident ray is perpendicular to the front face of the prism, the refracting angle, ς, becomes equal to β, Fig. 34, so that the equation is simplified to:

$$n = \frac{\sin(\delta + \beta)}{\sin \beta}$$ and may be given the form:

$$\tan \beta = \frac{\sin \delta}{n - \cos \delta}, \quad \text{XIV.}$$

when it is desired to determine the angle of the prism for any known value of the angle of deviation. If the refracting angle, ς, of a prism is very small, and the prism is placed in the position of minimum deviation, Fig. 33, then $\delta = (n - 1)\varsigma$. In other words, when $n = 1.5$, $\delta = \varsigma/2$.

However, this useful equation is frequently misapplied in ophthalmic literature, through neglect of its restrictions, and even though the deviation produced by a weak prism is about one-half of the apical angle for a prism that is *not exactly* exposed to minimum deviation, the equation is not applicable to *oblique* incident pencils of light. Ophthalmic prisms were originally numbered by the values of their refracting angles in degrees, and Dr. Edward Jackson was the first ophthalmologist to suggest that prisms should be designated according to their minimum deviation. However, as this would have entailed a very tedious and difficult method of measurement, by means of an expensive instrument, the spectrometer, requiring a degree of accuracy in manipulation and a knowledge in the reading of verniers with which opticians could not readily be made familiar, the writer, in 1890, suggested a novel nomenclature, based upon the tangent-deflection, dv, of 1 centimeter at the distance of one meter from the prism, Fig. 34, and which was subsequently adopted by the Ophthalmological Section of the American Medical Association in 1891. It seemed desirable to designate prisms according to their refractive powers, in order to bring them into harmony with the lenses used in ophthalmic practice; in fact, both lenses and prisms are now known by the work they perform, regardless of the refractive index and dimensions of the material used in their production. Therefore, the prism which produces a tangent-deflection of 1 cm. at one meter's distance was chosen as the unit of prismatic power.

In Fig. 35, for convenience of comparison, two overlapping principal sections of the prisms, ABC and abc, are shown with a common point of refraction at e. The prism, abc, by itself, produces a deflection, $dv = \delta$, in the meter-plane, PP_1; whereas, the prism, ABC, produces the same unit-deflection, $d_1v_1 = dv = \delta$, at one-half the distance, ½ meter, and therefore effects twice the unit-deflection at the meter-

plane, PP_1, *provided the incident rays of light are parallel.* Hence, the deflection produced by the refractive power is in inverse proportion to the distance at which the unit-deflection is produced, being fully in harmony with the refraction of a lens, which is in inverse proportion to the distance at which its image is formed. Thus, prisms, that effect say, two, three, or four times the unit-deflection in the meter-plane, will produce the same unit-deflection at one-half, one-third, or one-quarter of a meter, respectively. As prismatic refraction is a basic and inseparable part of lenticular refraction, the characteristic form and dimensions of the medium producing the former may be utterly ignored, so that it is merely necessary to specify the extent

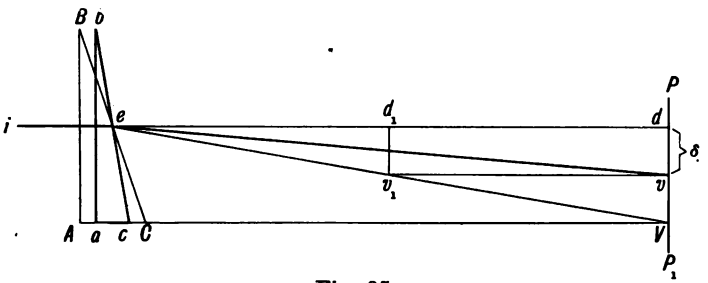

Fig. 35.

of the produced prismatic power. Regardless of the exceptions to be mentioned later, when the incident light proceeds from infinity, each ray that is incident upon the surface of a one-dioptry lens, one centimeter from its optic center, suffers the same centimeter-deflection toward its focus in the focal plane as that produced by the prismatic unit in the tangent-plane at the focal distance of the lens, one meter.

The Prism-Dioptry.

As prisms notably possess the property of apparently changing the position of objects seen through them, it was proposed in the new system that the apparent distance between the object and its *virtual* image, measured in a plane at right angles to the line of vision, should form the basis of comparison between the relative strengths of prisms. The tangent-deflection of one centimeter, *figuratively* located in a plane one meter from the prism, was, therefore, arbitrarily though befittingly chosen as the new unit of prismatic power, and was named the prism-dioptry. Moreover, as this *theoretic* tangent-deflection is based upon the assumption that *parallel* incident rays are to proceed from the object, it is necessary in practice to place the tangent-plane at a distance of 6 meters, and to correspondingly increase the unit-

deflection to 6 centimeters, when estimating the deviation produced by a prism while sighting through it at a suitable scale of equal parts, the prismometric scale. Therefore, in practice the *virtual* deflection is indirectly ascertained, through projection outward of the object's

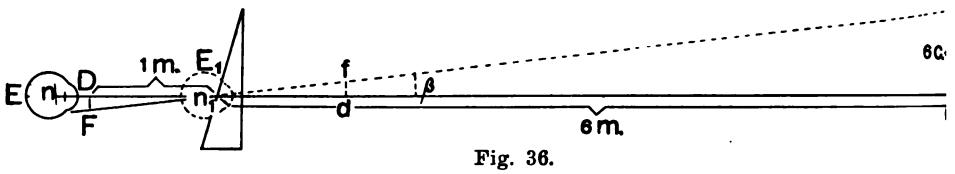

Fig. 36.

image within the eye, by looking through the prism, and noting the measurable apparent deflection of the image, f_1, from the object, d_1, *in the plane of the object six meters from the prism,* as shown in Fig. 36, where the actual deflection, DF, behind the prism, is equal to the virtual deflection, df, one meter in front of it.

Fig. 37.

The prismometric scale, Fig. 37, as devised by the author for the purpose of conveniently measuring the powers of prisms, consists of a series of parallel lines, separated by six-centimeter distances, successively marked from 0 to 10, with the line at zero longer than the rest, so that it may serve as the object to be viewed through the prism.

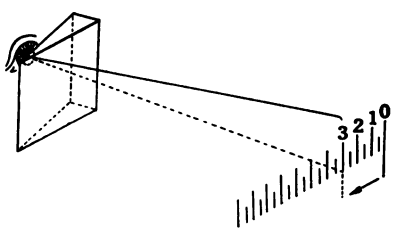

Fig. 38.

Commensurate subdivision of the scale facilitates readings to ½ or even ¼ prism-dioptries. In order to measure a prism, its base-apex line must occupy a horizontal position, with its anterior face parallel to and six meters from the plane of the scale. The upper edge of the prism should be on a level with the lower extremities of the scale-graduations, and the perpendicular apex-edge of the prism should be

directly opposite to the elongated index-line at zero, as illustrated in Fig. 38.

The observer places the pupil of his eye (its mate being closed) near the upper end of the apex-edge, so that the upper naked half of the pupil is exposed to the scale, while the ocular image of the index-line is seen through the prism and the lower half of the pupil. The observed apparent displacement of the index-line, as read from its apparent coincidence with one of the lines of the scale, determines the power of the prism in prism-dioptries, which, in Fig. 38, is shown to be 3 prism-dioptries.

In order that an object, d_1, Fig. 36, may be monocularly seen in its true position, the impulse to fixation so directs the visual axis that the object and the nodal point of the reduced eye, E, are in line, as drawn between n and d_1. As soon as a prism is interposed, the refraction at n_1 causes the pencil $d_1 n_1$ to assume the direction n_1F, which, with respect to the eye, E, does not enter its pupil, so that the eye will not only require to be brought nearer to admit the pencil n_1F, but also so close that its nodal point shall coincide with the point of refraction at n_1, if, indeed, the most accurate estimate of the deflection, $d_1 f_1$, is to be made. This, however, is practically impossible, wherefore, a slight error, due to parallax, will always be committed in reading the apparent deflection. Furthermore, readings should be taken as near the apex-edge as possible, in order to avoid another error that is due to increased thickness near the base, and which is explained in the section on *Astigmatism.*

The measurement of prisms, especially those of high power, should, therefore, be made with extreme care, by placing the pupil of the eye as closely as possible to the perpendicular apex-edge of the prism, Fig. 38. Prisms of high power, however, are not used as spectacle glasses, so that errors of a few prism-dioptries, more or less, are quite immaterial, when they are used merely to measure adduction and abduction, for instance. Measurement of the *theoretic* centimeter-deflection at the meter-distance is only justifiable when the rays proceeding from the observed index-line, at a distance of one meter, are rendered parallel by a carefully centered lens used as an eye-piece, or when the centimeter-deflection for incident parallel rays is projected upon a screen placed one meter from the prism. Both of these methods are, of course, less convenient. While it is true that a prism of 10 prism-dioptries produces a deflection that is 10 times greater than that produced by 1 prism-dioptry, yet two superposed 5-dioptral prisms will not produce a deflection exactly equal to 10 cm.; this being equally true of the resultant focal length of two superposed

5-dioptry lenses, which is not exactly 10 cm. In the former instance the discrepancy is due to the fact that twice the deflection is not produced by double the refracting angle; and in the second case, the increased thickness of the combined lenses is not a negligible dimension with respect to their resultant focal distance. Ever since 1894, American manufacturers have adopted the prism-dioptral system, and use the figure of a triangle as the symbolic sign for the prism-dioptry. Thus, one prism-dioptry ($1^{\triangle}$) is readily distinguished from the prism of one degree ($1°$) refracting angle, and, in fact, from prisms of any other system.*

As an exponent to the numerals of prismatic power the triangle originated with, and rightfully belongs to the prism-dioptry; and,

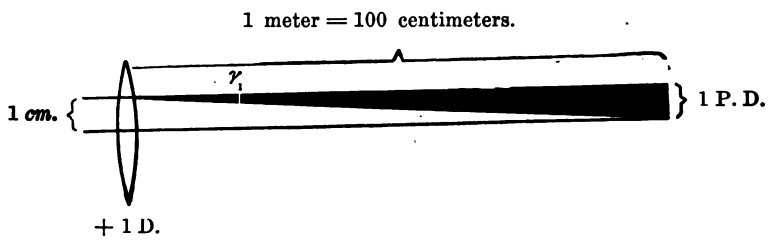

Fig. 39.

since this triangle represents both the principal section of a prism and the letter D of the old Greek alphabet, it is to be construed as an abbreviation symbolic of the words "prism" and "dioptry."

Moreover, ophthalmic prisms being frequently prescribed to counteract faulty deviations of the visual lines due to muscular imbalance, there is certainly need for an unequivocal symbol being punctiliously used by both ophthalmic practitioners and opticians.

Since lenses are a mere fusion of prisms of varying angles, the prism-dioptry may also be said to be the linear deflection which the emergent ray sustains at the focus of a meter-lens when the incident ray, parallel to the axis of the lens, impinges upon a peripheral part of the lens one centimeter from its optical center, Fig. 39.

* In order to be thoroughly consistent, in the interest of distinction the centrad should be designated by a figure containing an arc between two radii (1 ◁), since prisms made in America are marked (△) in prism-dioptries. Besides, Dr. W. S. Dennett, who suggested the centrad, had failed to disclose its possibilities with [illegible] to the decentration of lenses or the meter-angle, nor had he originally [illegible] to designate it. The centrad is equal to 0.01 of the radian, [illegible] tended at the center of a circle by an arc equal in length [illegible] the centrad is 1/100th part of the arc of 57° 17′ [illegible] ler arc-values, is only approximately equal to the

Therefore, the prism-dioptry also represents the measure of the angle of deviation, γ_1, for an eccentricity or decentration of one centimeter.

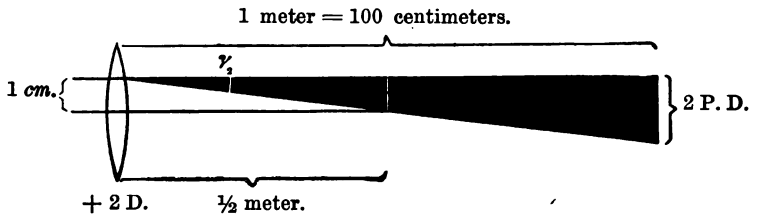

Fig. 40.

A ray impinging upon the same point of a 2-dioptry lens, Fig. 40, sustains the same unit-deflection at its focal distance, ½ meter, so that the measure of its angle of deviation, γ_2, is expressed by twice the deflection at the meter-plane, or 2 prism-dioptries. It will be later shown that a lens decentered twice or half as much will produce twice or half as many prism-dioptries as the lens possesses dioptries of lenticular refraction. The prism-dioptry is, therefore, but a sequence of the lens-dioptry, and, being based upon the proportion of 1:100, expresses a grade of angular inclination in daily use by engineers and scientists the world over. In order to reduce prism-dioptries to degrees of angular deviation, it is merely necessary to divide the prism-dioptries by 100, when the result will represent the tangents to correlative angles in degrees, and which are to be readily found in any table of natural tangents. Therefore, the prism-dioptry is virtually also a goniometric function, expressing the angle subtended by the tangent; and since different lenses, through varying decentrations, will produce different values of the angles of deviation, γ_1, γ_2, etc., it is possible, through the prism-dioptry, to determine the values of these angles in degrees, minutes, and seconds. Herein lies the unique simplicity and advantage of the prism-dioptry, especially when it is desired to know the exact angle of deviation produced by a prism of known dioptral power, or, conversely, when it is desired to ascertain the dioptral power of the prism that shall produce a given angle of deviation. The following examples will suffice to make this clear:

1. In the accompanying table, I, which has been abbreviatedly transcribed from a table of natural tangents, we find the tangent of 6°30′ to be 0.11394, for a radius of 1.

TABLE I.
NATURAL TANGENTS.

A.	TANGENTS.	D.1′
5°	0.08749	
		2.93
10′	0.09042	
		2.93
20′	0.09335	
		2.94
30′	0.09629	
		2.94
40′	0.09923	
		2.94
50′	0.10216	
		2.94
6°	0.10510	
		2.94
10′	0.10805	
		2.94
20′	0.11099	
		2.94
30′	0.11394	
		2.94
40′	0.11688	
		2.95
50′	0.11983	

TABLE II.
ANGLES OF DEVIATION AND PRISM-DIOPTRIES.

A.	
5°	8.749 = 100 × N.TG.
10′	9.042 = 100 × N.TG.
20′	9.335 = 100 × N.TG.
30′	9.629 = 100 × N.TG.
40′	9.923 = 100 × N.TG.
50′	10.216 = 100 × N.TG.
6°	10.510 = 100 × N.TG.
10′	10.805 = 100 × N.TG.
20′	11.099 = 100 × N.TG.
30′	11.394 = 100 × N.TG.
40′	11.688 = 100 × N.TG.
50′	11.983 = 100 × N.TG.

As the prism-dioptry is computed for a radius of 100 cm., we merely have to multiply the tangent given in this table by 100 to obtain 11.394$^{\Delta}$ as the prism that produces the desired angle of deviation, 6°30′. Hence, it is only necessary to displace the decimal points in the table, I, two points to the right in order to convert all of its tangent-values into prism-dioptries, when corresponding angles in the table will represent the angles of deviation produced by correlated prisms of the dioptral system, as shown in Table II.

Let us consider the following example, which differs only slightly from the preceding one:

2. Light being perpendicularly incident to one face of a prism, give the prism-dioptral power that produces a deviation of 6° 35′ 30″. It is apparent that the deviation in this case is only 5′ 30″, or 5.5′ greater than in the first example.

In the table, I, the tangent of 6°30′ = 0.11394, so that the tangent

sought must be one having a value between this and the tangent given in the table for 6°40′. In the table is given:

$$\tan 6° 40' = 0.11688$$
$$\tan 6° 30' = 0.11394$$

Hence, the difference for $10' = 0.00294$

Therefore, the difference for 1′ will be 1/10th of this, or 0.000294, and for 5.5′ must be multiplied by 5.5, which gives

	0.001617,	the difference for 5.5′.
This, added to	0.11394 ,	the tan 6° 30′,
gives:	0.11555	= tan 6° 35.5′ for a radius of 1.

The prism-dioptry radius being 100, we multiply this by 100, and obtain 11.555$^{\Delta}$ as the prism producing a deviation of 6° 35′ 30″. In some tables of natural tangents the differences of the tangent-values for 1′ are given in a separate column, as shown on the right in Table I, and which obviate the need for computation through subtraction of the tangent-values in the table in order to first ascertain the difference for 10′, and then for 1′.

This abbreviation is used in the following example:

3. Give the angle of deviation in degrees, minutes and seconds that is produced by a prism of 9 prism-dioptries when exposed to perpendicular incidence. We know that 9$^{\Delta}$ corresponds to a tangent-deflection of 9 cm. at 100 cm., or a tangent-value = 0.09.

The nearest tangent-value in the table is 0.08749, so that	0.09000
less the tan 5° =	0.08749
gives a difference in the tangent-value =	251.

In the table, I, between the tangents for 5° and 5° 10′, we find a difference of 29.3 is produced by 1′. Hence, we divide 251 by 29.3 in order to ascertain the number of minutes contained in the tangent-difference of 251, and find 8.56′. This must be added to 5°. Hence, $5° 8.56' = 5° 8' + 0.56 \times 60'' = 5° 8' 33.6''$.

Consequently 9$^{\Delta}$ produce an angular deviation of 5° 8′ 33″ +.

Therefore, it is evident that the mathematical accuracy of the prism-dioptry far exceeds the requirements of ophthalmic practice. Attention is also directed to the differences for 1′ shown in the table, I, and which do not in every instance exactly correspond with the differences that would be obtained by subtracting the lesser from the next

greater given tangent-value. This is due to the fact that the underscored figures in the fifth decimal place are one figure higher in this table than they would be in a table giving the tangent-values to the sixth or seventh decimal place, and from which the indicated differences for 1′ are taken and here given as an addendum to the table.

Incidentally, it may be stated that it is a scientific error to speak of prism-dioptries as degrees or centrads, for the reason that prism-dioptries are tangent-values and not degrees or parts of the radian. Carelessness of this kind leads only to bad habits of expression not usual among scientific men wishing to be technically understood. The student should bear in mind that optics is an exact science that is based upon a mathematical precision which does not admit of ambiguity of any kind.

The Relation of the Prism-Dioptry to the Lens-Dioptry.

As prisms are frequently combined with spheric lenses, it is here proposed to consider the relations of the prism-dioptry to such combinations, as well as to the equivalents which are to be obtained by a mere decentration of the spheric lenses themselves.

In the accompanying figure, 41, a thin lens is shown with its principal anterior and posterior foci, F_1 and F_2, equidistant from O, upon the optic axis, F_1OF_2.

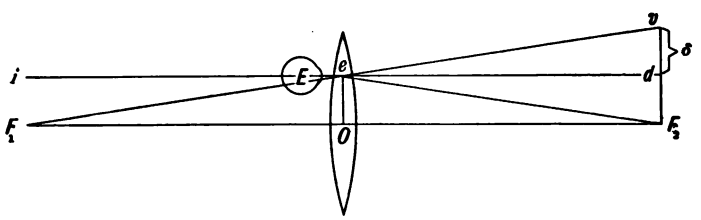

Fig. 41.

The ray, ie, which is parallel to the optic axis and incident at an eccentric point, e, of the lens, being refracted to the focal point, F_2, sustains a deflection, dF_2, *in the focal plane,* which is always equal to the decentration, Oe.

A ray, de, which is parallel to the optic axis and incident from the opposite direction, is refracted to the focal point, F_1, and, if received by the eye at E, is projected by it in the prolongation of F_1e to v.

Consequently, $dv = \delta$ is the measure of the apparent displacement of point, d, resulting from the prismatic action inherent at e. From ruent triangles, OeF_1, OeF_2, and dve, it follows that:

Oe.

Therefore, the tangent-deflection, δ, *at the focal plane of the lens,* is always equal to the amount of decentration, Oe, and is consequently in direct proportion to it. Augmented decentration of the lens will be associated with an increase in the prismatic action, resulting from a growing inclination of the tangents, $t_1 t_2 \ldots$, determining the obliquity of the spheric surface at corresponding opposite points of eccentricity, as shown in Fig. 42.

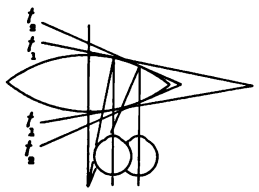

Fig. 42.

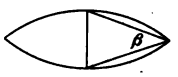

Fig. 43.

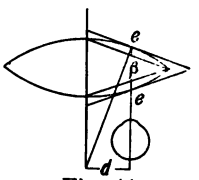

Fig. 44.

The inclination of these tangents, relatively to each other, becomes a maximum when the angle between them reaches 180°, that is to say, when they both coincide and so form the tangent to the lens which must then have become a perfect sphere. The amount of prismatic action it will be possible to obtain consequently depends only upon the diameter of the lens, yet, it will be shown that this has its limitations.

It is further obvious that a *virtual* prism of *constant* angle can not exist within the lens to produce the aforesaid variable results. The Fig. 43 is, therefore, apt to prove misleading, as there is but one fixed amount of decentration, d, Fig. 44, which corresponds to a prism of the angle, β, and which is determined by the tangents to the lens surfaces at e, there drawn parallel to the sides of the inscribed prism.

The principal focus of a spheric convex lens is known to be the point to which incident parallel pencils of light are made to converge after their passage through the lens. However, this is only true for pencils of light that are incident within a comparatively small area surrounding the pole upon the optic axis. Light that is incident outside of this limit does not reach the common focus, and produces what is termed spheric aberration; wherefore, the eye is provided with its iris, and every compound optical instrument with its diaphragm, in order to prevent entrance of the aberrative pencils that would otherwise interfere with a clear definition of the retinal image received through the comparatively small pupil.

In Fig. 45 a vertical section of a thin lens is shown, and upon which parallel rays, between o and b, are incident near the center of the lens,

with its principal focus, F, in the focal plane, DF, at a distance of one meter. The extreme pencil, AB, is shown *not* to reach the focus, F, therefore, produces aberration, and does not fall subject to the law of decentration hereinafter discussed. The outermost useful ray, ab, through refraction at b, suffers deflection in its original course from D to F; in other words, this tangent-deflection, DF, in the focal plane of any lens, can only be equal to the decentration, ob, when it is confined within the circumscribed non-aberrative area, ob, of the lens.

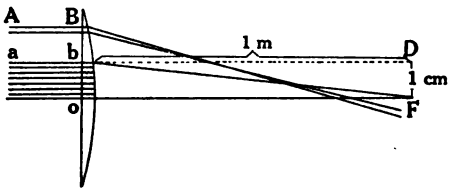

Fig. 45.

However, this limit is scarcely ever reached in the decentration of spectacle-lenses, whose size rarely admits of even a decentration of 5 millimeters; and yet, this is another instance in which the injunction respecting the practical application of the fundamental laws of optics must be respected, since the stronger the lens, the more restricted will be the area of decentration. Similarly, the generally accepted negligible thickness of lenses is known to assume such appreciable proportions in convex lenses of high power as to preclude their indiscriminate superposition, and for which reason their neutralization by concave lenses of equal power is not attainable.

Decentration.

Since the prism-dioptry is a deflection of 1 centimeter at one meter's distance, irrespective of the character of the refracting medium that

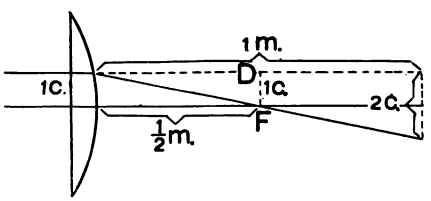

Fig. 46.

produces it, it follows that a decentration of one centimeter on a one-dioptry lens, Fig. 45, will produce a deflection, DF, in the focal plane at one meter, which is also equal to one centimeter and, therefore, one prism-dioptry.

Similarly, a two-dioptry lens, Fig. 46, decentered the same amount, produces a tangent-deflection equal to 1 centimeter *at its focus, half a meter,* which is equivalent to 2 centimeters in the meter-plane, or 2 prism-dioptries. Therefore, within the limitations of decentration above mentioned, the prismatic refraction of a decentered lens is also in inverse proportion to the distance at which the unit-deflection is produced. The following tabulation serves to make this clear:

Lens.	Decentration.	Tangent Deflection at the Focus.	Tangent Deflections in the Meter-plane.
1 D.	1 cm.	1 cm. at 1 meter	1 cm. = 1$^{\Delta}$
2 D.	1 cm.	1 cm. at ½ meter	2 cm. = 2$^{\Delta}$
3 D.	1 cm.	1 cm. at ⅓ meter	3 cm. = 3$^{\Delta}$

This table also reveals the author's unique law:

Any lens is capable of producing as many prism-dioptries as the lens possesses dioptries of refraction, provided it is decentered one centimeter.

On the other hand, the prism-dioptries will change as the decentration becomes greater or less, as shown in the following tabulation:

Lens.	Decentration in Centimeters.			Decentration in Millimeters.		
	1 cm.	2 cm.		1 mm.	3 mm.	6 mm.
0.25 D.	0.25	0.5	—Prism-Dioptries—	0.025	0.075	0.15
0.5 D.	0.5	1	" "	0.05	0.15	0.3
0.75 D.	0.75	1.5	" "	0.075	0.225	0.45
1 D.	1	2	" "	0.1	0.3	0.6
2 D.	2	4	" "	0.2	0.6	1.2

A lens of 2 D., limited by its size to a decentration of 3 mm., will afford 0.6$^{\Delta}$; whereas, a lens of 1 D., capable of a decentration of 6 mm., will produce the same prismatic effect, as shown above. In other words, a lens of one-half or one-third the power will require to be decentered twice or three times as much to secure the same number of prism-dioptries.

Therefore, it is only the size of the available lens which will, within the usual requirements of practice, set a limit to its prismatic power.

In Fig. 47, abc represents a vertical section of a 1 D. plano-convex lens, with three parallel rays, i_1, i_2, i_3, separated by one-centimeter distances, which are incident upon its plane side. These rays, after refraction, are collectively directed to the focal point, v, and there-

fore suffer perpendicular deflections in the focal plane *, dv, which are equal to the correlative decentrations of the rays, i_1, i_2, i_3, at their respective points of refraction.

As the spheric surface may be considered to be built up of an unbroken succession of infinitely small prisms of slightly varying angles, it is to be noted that the three chosen prisms, shown in their order of 1^{Δ}, 2^{Δ}, and 3^{Δ}, correspond to the respective decentrations of 1, 2 and 3 centimeters, and, therefore, produce correlative deflections in the focal plane, dv, *exactly* the same as the spheric surface at the same points of refraction. Some recent authors have failed to comprehend this unequivocal precision, wherefore, illustration of the principle is repeated in this form.

In Fig. 48 three concentric curvatures are shown to represent, respectively, the spheric or cylindric surfaces of 1, 2 and 4-dioptry

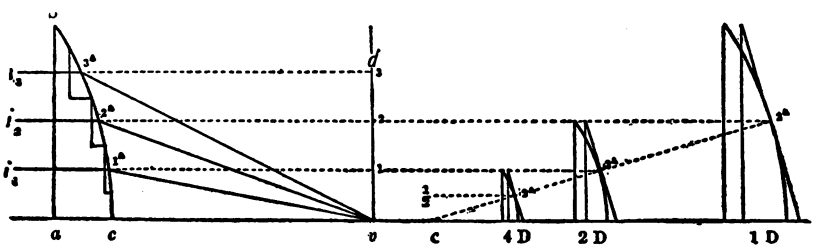

Fig. 47. Fig. 48.

lenses, in which the same prism of 2^{Δ} occupies a different position (decentration), relatively to the optic axis, on each of the lenses.

Beneath each section is given the dioptral power of the lens, which, being multiplied by the decentration in centimeters, shows the same prismatic power of 2^{Δ} to exist at a different though definite point on the surface of each lens.

Thus it is seen that every lens, whatever its dioptral power, contains all possible values of the prism-dioptry, which means that the prism-dioptry itself must constitute a distinct part of every lens of the dioptral system.

This is graphically demonstrated in the accompanying figures, in which the dimensions of decentration and lens-curvatures are materially exaggerated.

* "The great and enduring work of Gauss on the elucidation and simplification of optical laws has among its cardinal elements four planes—the anterior and posterior focal planes and the two principal planes (Hauptebenen); and the proportion of the size of image to object, as elucidated by the formula of Helmholtz, is calculated on the tangent plane. * * * This plane can, in the case of prism-deflection, be regarded in the same light as the focal plane of the standard lens."—*The Ophthalmic Review*, London, England, January, 1891.

In Fig. 49 the lens of 2 D., with a decentration $d_1 = 3$ mm., produces a deflection of 0.6$^\Delta$ *at the meter plane,* by reason of the obliquity of the lens-surfaces determined by the tangents at e_1, constituting a *virtual* prism of the angle, β, with its apex at a_1.

In Fig. 50 the lens of 1 D., whose radius $r_2 = 2r_1$, is decentered by the amount, $d_2 = 2d_1 = 6$ mm., to produce 0.6$^\Delta$ *at the meter plane*

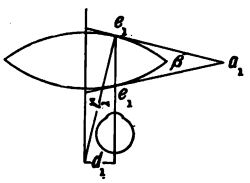

Fig. 49.

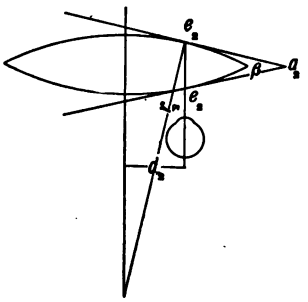

Fig. 50.

by a virtual prism of exactly the same angle, β, with its base at e_2e_2 and apex at a_2.

Consequently, it is only necessary to remember that:

The prism dioptries in decentered lenses are in direct proportion to their refraction and decentration.

Moreover, it is actually possible through decentration to determine the dioptral power of any pair of contra-generic lenses that effectually neutralize each other, and whose power may be *unknown.* All that is

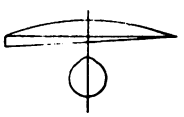

Fig. 51.

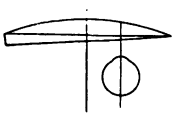

Fig. 52.

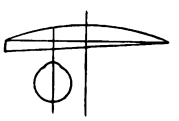

Fig. 53.

necessary is to place the lenses over each other, and to separate their optic centers exactly one centimeter, when the prism-dioptral power, as read from the prismometric scale, will be equal to the dioptral power of the lenses.

In cases where inadequate size of the lens prevents decentration, it becomes necessary to add to it a constant prism. Since the slightest decentration of a simple lens is certain to produce a corresponding prismatic effect, it is evident that the *constant* value of any prism. upon one of whose surfaces a spheric lens has been ground, will only

be retained when the optic axis of the lens-surface strictly coincides with the visual axis. See Fig. 51.

The effect of decentration will naturally be to increase or diminish the prismatic action of the constant prism which has been combined with the lens-surface.

Thus, the prism in Fig. 52 is increased by the prismatic action due to decentration of the lens, as the visual axis has been shifted toward its edge and the apex of the constant prism; whereas, in Fig. 53 it is decreased through decentration in the opposite direction.

Supposing a 5 D. lens to be combined with a prism of 2^{Δ}, the former being decentered 2 mm., through shifting the visual axis toward the apex of the prism. We know that a 5 D. lens produces 5^{Δ} when the decentered 1 cm., and, therefore, will produce 0.2 of 5^{Δ} when decentered 2 millimeters, which is equal to 1^{Δ}. Therefore, the constant prism of 2^{Δ} has been increased by 1^{Δ}, making it 3^{Δ}. A decentration of the lens to an equal amount in the opposite direction will leave but 1^{Δ} for the entire combination. Two millimeters have in this case affected the value of the constant prism by 50 per cent. of its active function, and which forcefully emphasizes the need for precision in centering lenses that are mounted in spectacles, especially when they are of considerable power.

The prism-dioptry and the meter-angle being directly dependent upon the inter-pupillary distance, it behooves us, in any endeavor to secure accurate results, to be exceedingly particular as to its measurement.

The Relation of the Prism-Dioptry to the Meter-Angle.

The unit-angle of convergence has been designated, by Nagel, the meter-angle; being the angle through which each eye turns when it abandons parallelism of the binocular lines of fixation in order to see an object situated upon the median line one meter from the eye.

Therefore, in Fig. 54 it is the arc embraced between the median line, MO, and the line of fixation, OE_1, whose length is one meter, so that

$$\sin \gamma_1 = \frac{b}{OE_1} = \frac{b}{C_1}$$

As the base-line, b, is a constant, one-half of the inter-pupillary distance of any individual, it follows that different values of the meter-angle (γ_1) are rendered solely dependent upon correspondingly varying values of C_1. Hence, if $C_1 = 1$ meter,

$$1\ \text{ma} = \text{arc sin}\frac{b}{C_1} = \text{arc sin}\frac{b}{1M} = b,$$

$$2\ \text{ma} = \text{arc sin}\frac{b}{C_2} = \text{arc sin}\frac{b}{\frac{1}{2}M} = 2\ b, \text{ and so on.}$$

ɔwever, the prism-dioptry differs from the meter-angle in that the stance of one meter is chosen parallel to the median line * instead

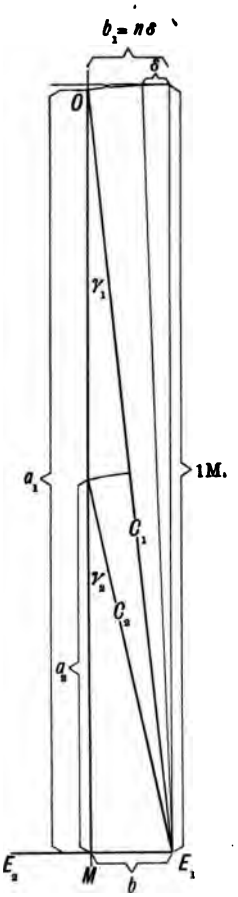

Fig. 54.

* Such substitution is admissible up to 5 meter-angles, as the difference ɜtween the tangent-deviation produced by the prism and the sine-value of five eter-angles amounts only to 4′ 42.7″.—A Metric System of Numbering and ɛasuring Prisms, Charles F. Prentice, *Archives of Ophthalmology,* Vol. XIX, ɜ90.

of upon the visual line, so that the definite dimension of one centimeter becomes the unit of comparison with respect to the base-line, which is known to be variable for different individuals. This is indicated in the diagram, Fig. 54, wherein $\delta = 1$ cm., the unit for the prism-dioptry; whereas, one meter-angle is indicated by $b = b_1 = n\,\delta$. Consequently, the meter-angle is n times greater than the prism-dioptry; or, the number of prism-dioptries,

$$n = \frac{b}{1 \text{ cm.}}.$$

Therefore, it is evident that the meter-angle contains as many prism-dioptries as there are centimeters in the base-line. Thus, for instance, for an inter-pupillary distance of 6 cm., the base-line will be 3 cm., so that

$$1 \text{ ma} = \frac{3 \text{ cm.}}{1 \text{ cm.}} = 3^{\Delta}.$$

Similarly, for an inter-pupillary distance of 5 centimeters, the base-line being equal to 2.5 centimeters, the meter-angle is equal to 2.5^{Δ}.

Therefore, for each inter-pupillary distance, a different prism is found necessary to substitute the meter-angle. This is quite natural, since greater demands for convergence are made necessary in wide than in narrow inter-pupillary distances. It is consequently only necessary to memorize the author's simple rule:

Read the subject's inter-pupillary distance in centimeters, when half of it will indicate the prism-dioptries required to substitute one meter-angle of convergence for each eye.

Or, to put it more uniquely:

Every individual carries his own meter-angle expressed in prism-dioptries between the centers of his ocular pupils and the crest of his nose.

Applications of the Prism-Dioptry.

The practical advantages of the prism-dioptry are best illustrated through citation of a few instances in which it is successfully applied. For instance:

1. What refracting medium should be placed before each of a pair of normal eyes having an inter-pupillary distance of 6 centimeters, in order that their powers of accommodation and convergence may *not*

be exercised in obtaining a distinct view of an object placed on the median line at ⅓ of a meter from the eyes?

Reference being had to Fig. 55, it is evident that a + 3 D. spheric lens will project parallel pencils into each eye from a point, D, ⅓ of a meter directly in front of it, thus obviating accommodation for the point, D, at this distance.

To suspend convergence, however, it will be necessary also to simultaneously project parallel pencils into both eyes from the single point, F. In order that rays divergent from F may be rendered parallel in passing out of the lens immediately in front of each eye, they must suffer a deflection, DF, equal to one-half of the inter-pupillary distance, 3 cm., in the plane, DD, at ⅓ of a meter, and therefore 9 centimeters in the meter-plane, which is 9$^{\Delta}$. Consequently, a + 3 D. spheric lens combined with a prism of 9$^{\Delta}$, with its base in before each eye, fulfills the requirements.

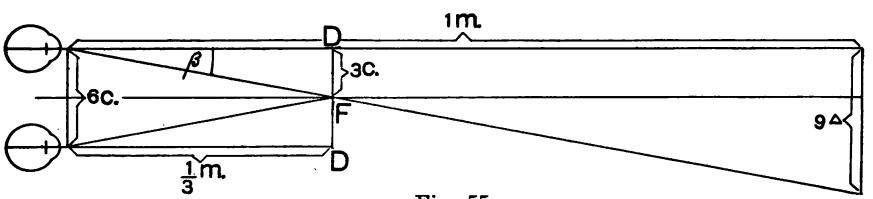

Fig. 55.

2. Two + 2 D. lenses in spectacles are found to be vertically decentered in opposite directions, one center being 5 mm. above, and the other 5 mm. below the inter-pupillary plane. The lenses are worn for both distance and reading by a young person known to have perfect balance of the extrinsic ocular muscles. What is the effect produced, and its extent?

A 5 mm. decentration of each lens is equivalent to a one-centimeter decentration of one of the lenses, provided its mate were properly centered, so that vertical diplopia would be produced through the decentered lens projecting the image within the eye at a different elevation with respect to the other.

The centimeter-decentration of a 2 D. lens produces a deviation corresponding to 2$^{\Delta}$ at the meter-plane, and, therefore, a vertical displacement of 12 cm. at 6 meters. Consequently, the object seen binocularly at this distance will appear as two objects whose centers are 12 cm. apart; whereas, in reading at ⅓ of a meter, 2$^{\Delta}$ produce a displacement equal to ⅓ of 2 cm., or 6⅔ mm., so that a line of type will appear as two lines of type separated by this distance.

3. A person claims to see an object placed at 6 meters' distance

as two objects, and indicates that their measured distance apa
40.2 centimeters. What is the angular deviation between his vi
axes?

40.2 cm. at 6 meters is equivalent to 1/6 of 40.2 cm. at one mete
6.7 cm. Therefore, 6.7/100 or 0.067 is the tangent of the angl
deviation. In the table of natural tangents this value correspon
3° 50′.

4. Being given two thin contra-generic lenses that are foun
exactly neutralize each other, yet, whose power is unknown, it i
quired to determine their power by the prismometric scale as the
implement at hand.

This is accomplished by accurately locating the optic center of
lens with an ink-dot, and upon one of them locating another simi
marked point exactly one centimeter from the center. Ther
superposing the lenses so that the center of one lens accurately c
the decentered centimeter-point of the other, and holding the
bined lenses before the eye, we read, through the contiguous u
edges of them, the prism-dioptral power upon the scale. The la
decentration teaches that the lens-dioptral power is equal to the p
dioptral power when a lens is decentered one centimeter, so th
the lenses in question produced, for instance, a five prism-di
deflection, the lenses must then also have five dioptries of lenti
refraction. Subsequent neutralization would prove this to be the
and would, therefore, also substantiate the author's law of dece
tion. The higher the power of the lenses, the more subtle bec
the experiment.

Whenever the prismometric scale is used to measure the p
dioptral power of any lens, it is obvious that a contra-generic lens
be used to counteract the lens-dioptral power of the lens whose
matic power alone it is sought to measure. In fact, this is no
only reliable method by which the prismatic power of any comp
lens involving prescribed power of this kind can be determined.
stances frequently arise in ophthalmic practice in which it be
necessary to add more prismatic power to a lens than can be
veniently obtained through decentration alone, this being accomp
through grinding either one or both lenticular surfaces upon o
both faces of a prism and, therefore, making a variety of such
binations possible. Combinations of spheric surfaces with prism
called *sphero prismatic lenses*, or *prismo-spheres*, and, being less
plicated than some others, are here chosen to explain the proc
necessary to measure their prism-dioptral power. For instance
5. In prismo-sphere, — 3 D. sph. ⌒ 2^, it i

necessary to carefully locate the optic center of a concave spheric lens of 3 dioptries, by marking it with an accurately placed and very small ink-dot, which, when perfectly dry, should be placed in contact with the spheric side of the prismo-sphere, as shown in figures 56 and 57. The operator next places the entire combination before one eye at exactly six meters from the scale, taking the usual precaution to have the base-apex line of the prismo-sphere horizontal, with its base to the left, and in such manner that the upper edge of the entire combination covers the lower half of the pupil. The index-line observed through the lenses will then appear to be displaced towards the right, relatively to the graduations seen through the uncovered upper half of the pupil, while that of the other eye is occluded. In the event of the index-line appearing to be displaced more or less than the required graduation marked "2," the operator needs only to shift the neutralizing lens carefully to the left or right, until the index-line exactly coincides with the second graduation. Care should be exercised not to change the position of the prismo-sphere in any way during this act, and, while in this position, an ink-dot should be placed upon the face of the prismo-sphere next to the eye, and precisely opposite to the dot on the neutralizing lens. The dot on the prismo-sphere then indicates the point which should become the center of the glass in the spectacle-frame.

The reason for this will be obvious from the following considerations:

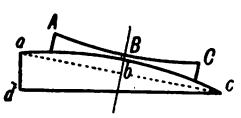

Fig. 56.

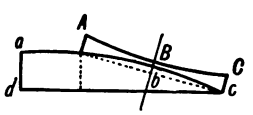

Fig. 57.

The concave lens, ABC, in Fig. 56, with its center at B, merely neutralizes the spheric power of the plano-convex lens, abc, thus leaving only the measurable effect of a prism, acd, just at the opposite points, Bb. By shifting the neutralizing lens laterally, as indicated in Fig. 57, the effect of a prism of greater angle is obtained.

Consequently, it is possible, within reasonable limits, by this means to correct any inaccuracy that may have been produced in the practical operation of grinding. The method is also applicable to *cylindro-prismatic* and *sphero-cylindro-prismatic lenses,* and in these cases, through neutralizing the cylindric element by an additional and carefully adjusted contra-generic cylinder, although this is naturally a little more difficult. Opticians who keep sphero-cylindric lenses in

stock will generally find it more convenient to use these in neutralizing compound lenses involving both cylindric and prismatic power. It is obvious that it will be much easier to hold and shift a neutralizing lens consisting of only one piece of glass; besides, it will lessen liability to error. In shifting compound neutralizing lenses, great care should be exercised to keep the cylindric axis of the neutralizing lens parallel to the cylindric axis of the lens whose prismatic power is to be measured, in case displacement of the lens-centers becomes necessary.

6. While it is not strictly within the province of an essay on ophthalmic lenses to include a discourse on the various methods employed in applying them to anomalies of vision, yet, in this instance an exception is thought to be fully justified in behalf of the prism-dioptry, especially as its useful applications by the ophthalmic practitioner are not published in other works.

For instance, in making a test for hyperphoria in the consultation room, the author's phorometric chart has been devised to enable the patient to exactly indicate the vertical distance apart of the dual lights, of a single light in the object-space, which he thus conceives through the two ocular images produced by any manifest deviation between his visual axes.

Admitting, by way of illustration, that the patient has decided them to be vertically six centimeters apart, which, being equivalent to 1 cm. at 1 meter's distance, indicates that the manifest vertical deviation between the visual axes is equal to 1^{Δ}. Therefore, the diagnostician is enabled promptly to decide that a prism of 1^{Δ}, placed with its base-apex line vertical, should correct the patient's *manifest* hyperphoria. Should further deviation persist, after this prism is introduced, it would merely suggest that there is more hyperphoria latent. However, the author's experience leads to the conclusion that an attempt should at first be made to correct *only the manifest hyperphoria.*

In connection with the chart the author preferably uses a $+12$ D. cylindric lens, which produces a much heavier and more readily recognized line of light than that observed through the Maddox rod.

The chart is shown in Fig. 58 as a blackboard 20 inches square, having upon its surface eight vertically and eight horizontally arranged dots, which are separated by 6-centimeter distances, so that each interval of space between the dots represents 1^{Δ} at 6 meters' distance from the eye. Charts designed to be used at a lesser distance than 6 meters are unreliable and incorrect from a scientific point of view. See sections on *The Prism-dioptry* and *Astigmatism.* A light

is placed behind a piece of red glass in a circular opening in the center of the board.

The patient being directed to look through his *distance* glasses at the central light with both eyes—while the cylindric lens is placed by the operator, first vertically and then horizontally before one eye —will be able to promptly indicate any displacement of the red line of light from the center, by stating through which of the dots the red line seems to pass.

For instance, should the red line of light appear to pass horizontally through the second dot above the center, when the cylindric lens is properly placed before the patient's right eye, the operator will at once decide that a prism of 2^{Δ}, placed with its base up before the patient's right eye, should cause the red line to drop two points to the

Fig. 58.

center, thus counteracting the manifest vertical deviation of his visual axes.

It is, of course, of the utmost importance that the centers of the distance glasses worn during the test should be carefully adjusted in respect to their inter-pupillary distance and elevation. When the vertical deviation of the visual axes exceeds the amount provided for by the chart (4^{Δ}), correction by prisms is more or less uncertain, owing to the phenomena of internal reflection and spectral color more or less apparent in prisms of higher power.

Lateral deviations of the visual axes are to be similarly determined by means of the horizontal dots on the board, though great caution should be exercised in reaching conclusions respecting them. Besides, in such cases prismatic corrections of sufficient power to be effective are both cumbersome and generally unsatisfactory.

The chart with its rod-image is unquestionably more reliable than any method in which a prism, producing either intended vertical or lateral displacement of the ocular image, is substituted for the rod in an attempt to determine the *actual* direction of the visual lines at

right angles to the artificial displacement first induced by such prism. It is evident that the direction of the visual line, whatever it may be, is due to a force, the *resultant of a system of component forces* (Ganot's "*Physics,*" or any book of engineering formulæ) exercised by the muscles attached to each eye-ball. Consequently, in accordance with the established physical law, to apply a prism that shall induce diplopia in any direction is equivalent to adding another force to the system which must then have a different resultant. In any event, the reliability of this sort of test certainly depends upon whether or not the relative strengths of the muscles remain unchanged by the addition of the prism. That a prism does stimulate a force foreign to the normal action of a muscle is a fact supported by long established experience. For instance, it is well known that the powers of adduction and abduction are to be measured by a prism placed with its apex toward the muscle to be taxed, and which shows that a limit of endurance, with respect to the power of the prism, can be reached. Furthermore, where muscular imbalance really exists, it is logical to suppose that a prism of sufficient power to produce manifest diplopia will be more apt to cause a change from the true direction of the visual lines than if such an anomaly did not exist.

Therefore, the conclusion is justified that the *actual* directions of the visual lines that are sought to be determined are not the same as the directions that are made apparent by use of the diplopia-producing prism, so that reliable information respecting the *phorias* is not to be secured through this means.

7. As a matter of further interest to the diagnostician it may be stated that the author quite recently established the law:

In manifest hyperphoria of one prism-dioptry the distance between the chiasmal image-centers is equal to one hundredth part of the distance between the nodal point and the retina in the deviating eye (*Ophthalmic Record,* Chicago, Ill., February, 1914). See definition, **Chiasmal image.**

In fact, for Donders' reduced eye, whose first focal length $F' = 15$ mm., it was demonstrated that a deviation of 1^{Δ} between the visual axes produced a foveal displacement in the deviating eye equal to 0.15 mm., this also being equal to the binocularly produced separation of the chiasmal image-centers in case either contiguous or overlapping images exist.

Moreover, it was shown that the figurative chiasmal images are vertically *separated* when the diameter of the object viewed binocularly is less than the prism-dioptral deviation, and that they overlap when the diameter of the object is greater than the prism-dioptral devia-

tion between the visual axes. The limits between which *overlapped* images are produced are made apparent in the following necessarily exaggerated diagrams, Figures 59, 60 and 61, and in which the dimensions of the various retinal images have been calculated for white square targets of different sizes placed at 6 meters' distance from the eyes.

DEVIATION BETWEEN THE VISUAL AXES = 1 PRISM-DIOPTRY.

Foveal Displacement, Separation of the Chiasmal Image-Centers and the Image-Extension = d = 0.15 mm. for Donders' Reduced Eye.

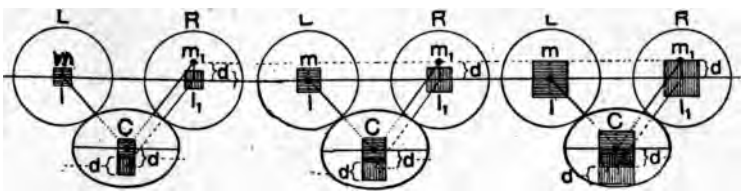

Fig. 59.	Fig. 60.	Fig. 61.
Object, 60 mm. square.	Object, 87.3 mm. square.	Object, 120 mm. square.
Image, 0.15 mm. square.	Image, 0.2185 mm. square.	Image, 0.3 mm. square.

This is of special interest, because it familiarizes the ophthalmic practitioner with the fact that all of the letters of the visual test-card that are smaller than 60 mm. square, at 6 meters from the reduced eye, produce separated chiasmal images for a deviation of 1^{Δ} between the visual axes; whereas, larger letters produce overlapped images.

Of course, with increased deviation between the visual axes, or increased distance (F′) between the nodal point and the fovea of the deviating eye, there will be a proportionate increase in the separation of the chiasmal image-centers.

Thus: 1^{Δ}, 2^{Δ}, 3^{Δ}, 4^{Δ} of deviation between the visual axes, produce 0.15, 0.3, 0.45, 0.6 mm. separation of the chiasmal image-centers, when F′ = 15 mm., or 0.16, 0.32, 0.48, 0.64 mm. separation of the chiasmal image-centers, when F′ = 16 mm.

In short, when the diameter of the object at 6 meters' distance is exactly equal to the prism-dioptral deviation between the visual axes, contiguous chiasmal images are formed whose line of contact is the boundary between separated and overlapped images.

Familiarity with the principles herein set forth should not only impress the ophthalmic surgeon with the extreme delicacy of the muscular balance necessary to maintain orthoscopic binocular vision, but also encourage him to make effort to counteract minor degrees of hyperphoria through optical rather than by surgical means; especially when it is emphasized that the displacement of the fovea in the

deviating eye amounts only to 0.6 of a millimeter for a deviation between the visual axes of 4 prism-dioptries, commonly thought to be effectually corrigible through the application of surgical skill, and which it is conceded bespeaks marvelous dexterity where it has been successfully applied in cases of this kind.

However, in order to avoid misconception, it is emphasized that the chiasmal image is not to be construed as an actual image, but that "it shall signify (figuratively, of course) that orderly assemblage of the optic nerve-fibrils which receive their individual stimuli from corresponding points in each retinal image" (*Ophthalmic Record,* Chicago, Ill., July, 1914); this figure of speech being used to make it clear that the drawings are produced through projection of corresponding points within the retinal image-areas into the chiasmal field, notwithstanding the well known fact that the latter's diameter is inadequate to accommodate the implied arrangement of the nerve-fibrils that are supposed to transmit, in one way or another, conjecturally or otherwise, the ocular images themselves. In fact, the diagrams show the needful exaggeration of the chiasmal field, which is used merely as an imaginary transverse plane in which to actually picture the retinal images as they are mentally conceived, and which, in this supposed state of perception, are *figuratively* called chiasmal images, in order to distinguish them from the actual retinal images that occupy differently located areas. In fact, this and the advantage the chiasmal image offers to the draftsman in producing lucid diagrams constitute the only excuse offered for its use.

Moreover, the precedent established by Professors Hall and Hartwell, with respect to *corresponding points,* is accepted by assuming that the axial image-point of each eye at the fovea is conveyed by its corresponding optic nerve-fibril to the center of the chiasmal field; an assumption actually made necessary to reconcile mental orientation of the image with respect to a common center of visual perception of an object centered upon the median line, as well as the construction of the diagrams portraying them. Consequently, it may be safely conceded that, if both macular centers were not transmitted by their own nerve-fibrils so as to produce superposed images of themselves at the center of the optic commissure, there not being any substantiated evidence to the contrary, and a similar coalition of them did not exist somewhere in the brain itself, the images in both eyes would necessarily be conceived by the brain as dual images, so that diplopia would exist under all circumstances. It is, therefore, obvious that, in orthophoria the centers of the ocular images must be so projected, at least in a drawing, as to exactly cover each other at the center of the

figurative chiasmal field; or, expressed in terms of the printer, they must be in perfect register, thus requiring their point-to-point projection to this center * in the diagram in order to correspond to the supposed center of orientation for the orthoscopically conceived brain-image itself. Such a point of orientation is quite as necessary to pictorial illustration in ophthalmology as the prime meridian at Green-

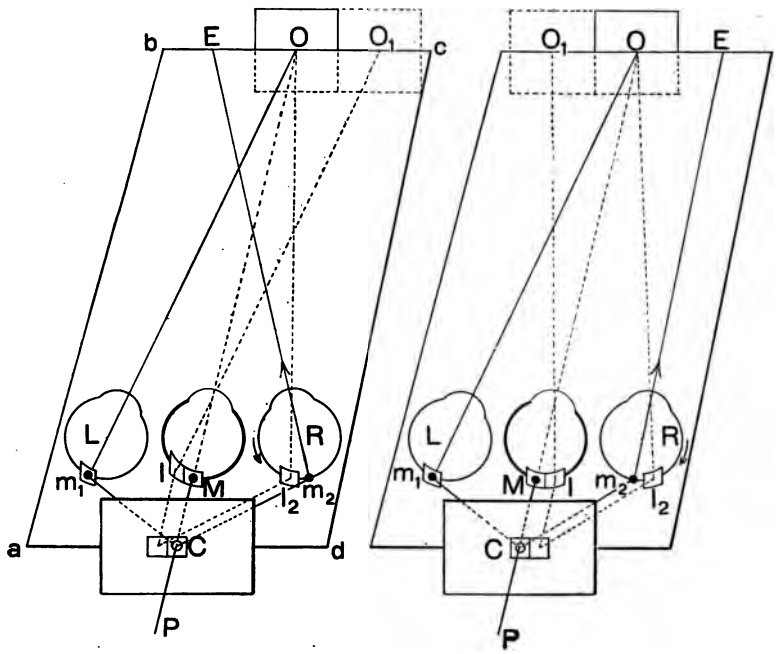

Fig. 62.
Homonymous Projection.

Fig. 63.
Heteronymous Projection.

Contiguous Mean Cyclopean Images derived from Figurative Chiasmal Images.

wich is to navigation, or the pole star is to astronomy, for without it one could not definitely interpret the difference between right and left, or up and down, in constructing a diagram in accordance with the rules of projection that apply to mechanical drawing. Consequently, so far as pictorial illustration is concerned, it is necessary to accept this point-to-point correspondence, even though the most recent investigations may have proven that this is not punctiliously correct for certain peripheral extensions of the object from its center on the median line, or for points of binocular fixation not located upon it.

* This correspondence of macular points in the superposed retinæ is accepted by Hall, Hartwell and Titchener.

These considerations make it evident that an image to be *orthoscopically* conceived by the brain must be produced through fusion into a single image, somewhere and by some means, of two similarly conceived ocular images of the same size. This being the fact, the single image, located at the visual center of the brain, is produced quite the same as if it were transmitted from a single image whose center is the *point of orientation* at the fovea in a single *mean eye* (cyclops),* situated on the median line, exactly between the eyes. See Fig. 62. Therefore, *a figurative cyclopean fovea,* in addition to the *center of the chiasmal field,* may be made to serve as another point of orientation in comparing the relative positions of the retinal images in the phorias.

In order to demonstrate the purpose and need of at least one point of orientation, although two different ones will be here jointly applied, let it be supposed that the diagram, Fig. 62, represents a horizontal plane, abcd, in which the corresponding sections of the right eye, R, and the left eye, L, are located to view the object, O, upon the median line, MO. It is also assumed that the visual axis of the right eye, R, is faultily directed towards E, as in esophoria, so that its macula, m_2, is turned to the right; whereas, the macula m_1, of the left eye, L, retains its normal position with respect to the object, O; and, therefore, also with respect to the center, C, of the chiasmal field and the macula, M, of the *mean eye* on the median line, and to which points of orientation the *macular center,* m_2, in the right eye is also projected. Consequently, the macular centers, m_1, and m_2, in both eyes have the same points of orientation, C and M, in common, while the

* "It makes a difference in apparent projection-distance whether an object be seen with the right or the left eye, so that we ought not to say identical points with reference to an object, but *coincident points,* as if the retinæ were *laid one within the other like two cups;* etc., etc.; that the error in putting the finger through a ring is greater when the right than when the left eye is closed, so that a *true, mean, cyclopean eye* would be slightly to the right of the median line." Bilateral Asymmetry of Function. By G. Stanley Hall and E. M. Hartwell. The Psychophysical Laboratory, Johns Hopkins University, Baltimore, Md.—*Mind,* Vol. IX, p. 93, London, 1884.

The following extract from Titchener's *Textbook of Psychology* (1911, p. 309) corroborates the character of a mean eye, in which "a pin" is substituted for the median line: "Think of the two retinas as slipped, the one over the other, and as held together by a pin driven through the superimposed foveas. *The two pin-holes then represent corresponding points,* the retinal points *stimulated by the point in objective space which the eyes,* at any given moment, *are fixating.* Let other pins be driven vertically through the two retinas, at any points round about the fovea: in the rough, every pair of holes will represent a pair of corresponding points. Now, it is clear, if you work the matter out by help of diagrams, that when the eyes are in a certain fixed position, only a certain number of the points in objective space can be imaged upon corresponding points."

The italics have been introduced by the writer in order to emphasize evident corroboration of his contention.

image-center, m_1, *of the left eye alone* is transmitted to these points.

But, the center of the image, I_2, projected from O into the right eye, is situated on the left side of its macula, m_2, and is, therefore, transmitted with equal displacement so as to be located on the left side of the centers of orientation, C and M, in the chiasmal field and mean eye, respectively. Therefore, the displaced image, I_2, in the *right eye,* is transmitted to and located on the *left side* of the mean eye as the false cyclopean image, I; and it is this image, belonging to the right eye, that is homonymously projected to O_1, on the right side of and at the same prism-dioptral distance, EO, from the object O.

The points m_1, m_2, C and M are corresponding points with reference to the axis, PCM, of bilateral symmetry within the cranium, which coincides with the median line, MO, in the object-space and is directed to the supposed center, P, of image-perception in the brain.

In Fig. 63, heteronymous projection of the ocular images is illustrated. Both diagrams show that the chiasmal and mean cyclopean images, respectively, are contiguous, because the horizontal diameter of the object is made equal to the prism-dioptral deflection EO, so that the real object, O, and the mentally conceived *apparent object,* O_1, are also in contact.

The figurative mean cyclopean images, I and M, are graphically projected from their corresponding chiasmal images, so that *conjointly* they make it possible to pictorially illustrate either honomymous or heteronymous diplopia in a manner not hitherto lucidly accomplished.

Consequently, the *nodal point* of the "mean cyclopean eye," mentioned in 1884 by Hall and Hartwell (who more consistently used the median line instead of "the pin driven through the superimposed foveas" mentioned by Titchener) serves, in the writer's drawings, as the pole for the orientation, in any meridian, of the *directions of image-projection into the object-space;* whereas, the *chiasmal center* is the point for the orientation of the *distance between the transmitted dual ocular images,* in case separation or overlapping of them exists. Therefore, it is obvious that both of these imaginary points of orientation may be made to serve as a useful *working hypothesis* in making a lucid drawing in physiologic optics, while the *chiasmal image* specifically commends itself as a *figure of speech,* when it is desired to differentiate between the ocular images and their corresponding brain-images, whose actual locations and physical existence have not yet been definitely determined. In fact, "brain-image" is also only a figure of speech, since it is just as imaginary as its correlative chiasmal image, which, at least figuratively, occupies a definite location.

In view of the indeterminate psychophysical character of these images, and an effort being made to picture them in a drawing, the delineator will at least need to assume that the center, P, of the so-called brain-image is the center of image-perception; that it is located in the horizontal plane, in juxtaposition to the center, M, of the centered cyclopean image of the object on the median line and, therefore, coincident with the center of the chiasmal image; said line, PCM, *within the cranium,* being considered the axis of visual orientation, or the directrix of bilateral symmetry of vision in the mind of the draftsman at least, if not in that of the psychologist.

Thus the interdependence of the chiasmal center and the nodal point of the mean eye have been disclosed, and which, for the purpose of making a lucid and mathematically serviceable drawing in accordance with the rules of projection, must be considered as being located in planes that are at right angles to each other.

8. It being within the province of the ophthalmic practitioner to discriminate in his choice of the means that shall be employed by the optician in practically executing prescriptions for glasses, a few instances are here given in which such discrimination is both convenient and profitable. The following examples are confined to hyperphoria, because it may be more effectually counteracted through the use of prisms than any other muscular imbalance.

Assuming that the diagnosis has resulted in the prescription:

O.D. + 3 D. sph. $\supset$ 2^{Δ} base up,
O.S. + 3 D. sph.

The optician would execute this prescription literally, and, therefore, grind a + 3 D. spheric surface upon a prism of 2^{Δ} at objectionably increased cost, because the prism-dioptries required in the right lens are in excess of the prismatic power attainable through decentration of a stock-lens.

However, the discriminating practitioner will convert his office-data so as to read: O.D. + 3 D. sph. $\supset$ 1^{Δ} base up, and O.S. + 3 D. sph. $\supset$ 1^{Δ} base down, but, in writing the prescription, will take advantage of the law of decentration applied to a + 3 D. lens. In accordance with this law, 3 D. decentered 1 cm. gives 3^{Δ}, yet, as only 1^{Δ} is needed, a decentration of ⅓ cm. for each lens will satisfy the requirements. Therefore, the prescription should be written:

O.D. + 3 D. sph., decentered ⅓ cm. up,
O.S. + 3 D. sph., decentered ⅓ cm. down,

and by which it is understood that the optic center of the convex lens is above the visual axis of the right eye, and below it before the left eye. Fig. 64 indicates the position of the lens with respect to the right eye.

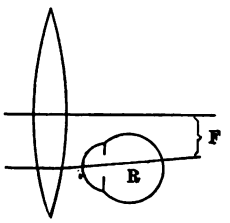

Fig. 64.

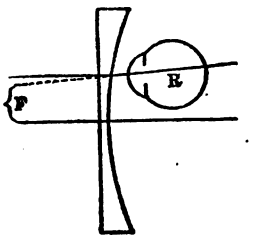

Fig. 65.

As the term "decentration" always signifies displacement of the lens relatively to the visual axis, it is to be remembered that the upturned base of a prism combined with a concave lens calls for the thick edge of the lens being placed upward, so that in such cases attempted decentration must be downward, Fig. 65. In other words, the decentration of concave lenses must be in the opposite direction to that of the specified base of the prism. For instance, as cylindric refraction is always perpendicular to the indicated axis:

9. When O.D. — 2 D. cyl., axis 180° ⌣ 2$^{\Delta}$ up, and O.S. — 2 D. cyl., axis 180° are written to call for decentration, the prescription should be:

O.D. — 2 D. cyl., axis 180°, decentered ½ cm. down,
O.S. — 2 D. cyl., axis 180°, decentered ½ cm. up.

The optician's ability to apply the law of decentration is governed entirely by the size of the lenses furnished by the manufacturers, who nevertheless now make extra large lenses at a slightly increased cost, and which admit of a 5 mm. decentration vertically, or 3 mm. horizontally, provided the corresponding diameters of the oval shapes required for the lenses are not too great.

10. As another example let us cite a case in which the examination results in O.D. + 2.75 D. sph. ⌣ + 1.25 cyl., axis 180° ⌣ 2$^{\Delta}$ base up, and O.S. + 4 D. sph.

In case the mounted lenses are to be of average size, the prescription may be written:

O.D. + 2.75 D. sph. ⌣ + 1.25 D. cyl., axis 180°, decentered 5 mm. up,
O.S. + 4 D. sph.;

or, if the mounted lenses are to be of large size:

O.D. + 2.75 D. sph. ⌒ + 1.25 D. cyl., axis 180°, decentered 2½ mm. up,
O.S. + 4 D. sph., decentered 2½ mm. down.

The above examples suffice to show how easily and with what absolute accuracy these prescriptions may be executed without incurring the additional expense of grinding prismatic combinations. In fact, this expense should only be incurred in those cases where decentration is impossible on account of an insufficient size of the lenses. A glance at the prescription will determine at once which of the methods to apply. Take, for example, a case like:

11. O.D. + 1 D. sph. ⌒ 1^{Δ} base up, and
O.S. + 1 D. cyl., axis 90° ⌒ 1^{Δ} base down.

In this case, as our commercial lenses are too small to bear a decentration of 1 cm., it would be necessary to grind the lenses as the prescription is written, though even here, to lessen the expense of the left lens, it is preferable to write:

O.D. + 1 D. sph. ⌒ 2^{Δ} base up,
O.S. + 1 D. cyl., axis 90°.

If these lenses are to be rimless eye glasses, effort should be made to match the lenses as nearly as possible in thickness, in order to avoid a preponderance of weight that would otherwise result in the right lens. Whenever prismatic corrections in the vertical direction are necessary, a preference should be given to place the base up before one eye rather than base down before the other, especially where the prism is stronger than 2^{Δ}. This is explained in the fact that the eyes are much more frequently turned downward than upward, and are, therefore, more exposed to the annoying internal reflections that are noticeable near the base of the prism when it is down. This, in part, also explains why strong prisms with their bases towards the nose are not satisfactory. See section on *Astigmatism*.

12. It is intended here to illustrate the principal defect which so frequently leads to disappointment in the use of cemented bifocal lenses (more specifically described in a subsequent section), as well as to explain how it may be obviated through application of the prism-dioptry. When occasion demands, it is common practice among ophthalmologists to prescribe glasses for "reading" and "distance," and occasionally with rather vague instructions to the optician to provide the necessary lenticular corrections in the form of bifocal lenses. These, in the event of their being of the so-called "cemented" variety,

the optician executes by cementing two thin lenticular segments to the lower surfaces of the lenses whose upper fields are used for distant vision; both of the segments being cut from the peripheral parts of that lens which produces the requisite amplifying or reducing power in the lenticular combination. The principal effort of the optician, at present, is to make this lens as thin as possible, and to reduce its diameter so as to enable him to secure at least two segments of sufficient size for the ocular fields required.

Economy and extreme thinness of the segment are doubtless desirable, but these are only of minor importance. Spectacles as now constructed, exclusively with this in view, are rarely ever free from a prismatic action, operative vertically, which renders them very uncomfortable to wear, and frequently useless. This is especially true in lenses of high power, *and in cases involving cylindric combinations,* where the spheric refraction is obtained by spheric curvature of *one*

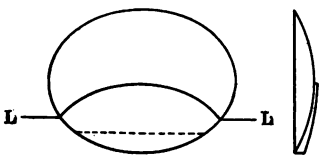

Fig. 66.

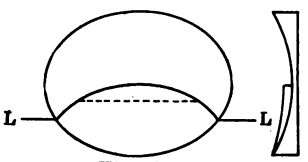

Fig. 67.

surface only. With a view to brevity, only the latter type of correction will be discussed.

The accompanying diagrams, Fig. 66 and Fig. 67, will serve to illustrate the defect referred to.

In each of these figures the line, LL, is drawn upon the paper, which is supposed to be placed several inches behind the bifocal spherocylindric lens. The line viewed through the lower segmental field appears disconnected, being deflected by the prismatic action resulting from decentration of the "distance" and "reading" lenses relatively to each other. It is, of course, customary to have the lower smaller field for reading, as shown above, but, for convenience of easier demonstration, the reader may make the interesting experiment of superposing two concave lenses, say, — 4.5 D. and — 2.5 D., on which the optic centers have been previously marked with ink-dots, and allowing them to occupy the positions shown in Fig. 68, in which the overlapping parts are in the smaller field for distance.

By more widely separating the lens-centers (designated by +), it will be observed that the disconnected portion of the line, LL, as seen through the lenses, will appear displaced to a lesser degree, and, if the upper concave lens, — 2.5 D., is chosen of sufficient diameter, it

will be possible to secure a distance between the lens-centers which will exhibit the line, LL, unbroken, as in Fig. 69. This shows that the prismatic action depends only upon the distance separating the lens-centers and, therefore, also that *the lens,* specifically in this case the upper one, from which the segmental wafers should be cut, *must have a definite diameter for every combination,* if the prismatic action is to be eliminated. We shall cite an example in which we have for distant vision: — 7 D. sph. ⌣ — 2 D. cyl., ax. 180, and for reading: — 4.5 D. sph. ⌣ — 2 D. cyl., ax. 180, which, executed as a cemented bifocal lens, calls for a + 2.5 D. periscopic segment. This leads us to the proposition:

What peripheral part of a 2.5 D. periscopic convex lens should be used as a segment to insure freedom from prismatic action in the center

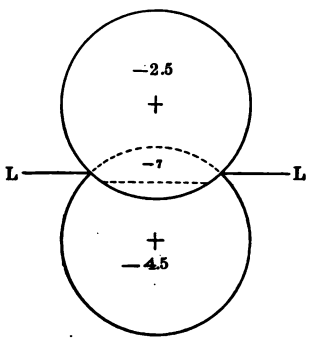

Fig. 68.

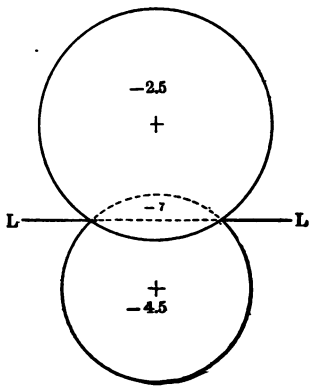

Fig. 69.

of the segment 7 millimeters below the center of the distance lens, — 7 D. sph. ⌣ — 2 D. cyl., ax. 180?

The key to its solution is to be found in the law that "a lens decentered one centimeter will produce as many prism-dioptries as the lens has dioptries of refraction."

The center of the segment being 7 millimeters (0.7 cm.) below the center of the distance lens, makes it obvious that we have a prismatic action at this point acting vertically, on account of the — 7 D. sph. and — 2 D. cyl., ax. 180, which is equivalent to a decentration of 0.7 cm. on 9 D. to be neutralized by the segment. Reverting to the stated law we find that 9 D. decentered 1 cm. affords 9^{Δ}, therefore, 0.7 cm. will give 0.7 of 9, or 6.3^{Δ} as the prismatic action to be overcome.

The segment of + 2.5 D. decentrated 1 cm. gives only 2.5^{Δ}, so that it takes a decentration of 2.5 cm. to produce $2.5 \times 2.5 = 6.25^{\Delta}$. Therefore, this segment, when placed with its thin edge at the lower edge

of the concave distance lens, will neutralize the existing 6.3$^{\Delta}$ with an error of only 0.05$^{\Delta}$. As will be later shown, the + 2.5 D. lens, in order to be large enough for so great a decentration, must be at least 64 millimeters in diameter, and should be ground to a knife-edge to insure extreme thinness. As this example came to the author's notice, the instructions given to the mechanic, who successfully executed the lenses, are here repeated as follows:

"Make a 2.5 D. periscopic convex lens (+ 7 D. on — 4.5 D.) 64 millimeters in diameter, worked to a knife-edge at the periphery, and, after marking its optic center, lay off four points, p, p, p, p, 25 millimeters from the center, as shown in the diagram, Fig. 70."

"Replace the lens on the *convex* grinding tool, and cut the lens through the indicated dotted transverse diameters into four equal

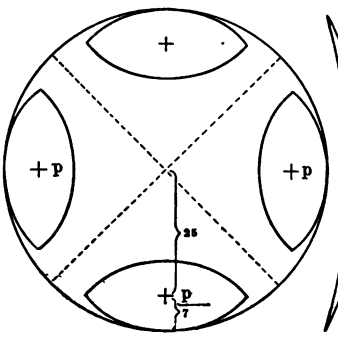

Fig. 70.

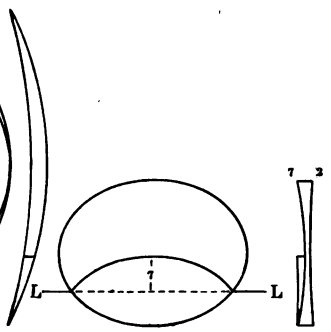

Fig. 71.

parts. This will secure four quadrants, with ample provision against accident. Select for both eyes two of the most perfect quadrants, and cut from them the peripheral segments to the shape indicated, and cement the segments into the concave 7 D. spheric surface of the sphero-cylindric lens, with their thin edges down, when a line, LL, viewed through the reading lenses, will appear continuous as in Fig. 71."

It is obvious that the diameter of the lens is determined by adding twice the decentration (25 × 2) to one full width of the wafer (7 × 2), which gives 64 millimeters.

In conclusion, it is believed to have been demonstrated that applications of the prism-dioptry do not call for profound mathematical knowledge, even though this optical unit is based upon accurate mathematical deductions that involve a principle which is easily understood and capable of being applied within the confining limits set by the fundamental laws of optical science. What could be more simple

than to remember the centimeter in its connection with the prism-dioptry, as we do the meter in its relation to the lens-dioptry?

The prism-dioptry commends itself because every lens, whatever its dioptral power, contains all possible values of the prism-dioptry, which means that it unavoidably constitutes a distinct part of every lens of the dioptral system. Therefore, the prism-dioptry stands unchallenged in its unique ability to harmonize all of the refracting elements in the ophthalmic lens-case, through establishing a scientifically accurate, direct and inseparable relationship between prisms and lenses, and which cannot be accomplished by any other advocated system without appropriating the laurels of the prism-dioptry.

Cylindric Lenses.

A cylindric surface is generated through rotation of a straight line that is parallel to the axis of revolution, Fig. 72, wherefore, the center of curvature of any principal section is on the axis, AA, the so-called axis of the cylindric lens.

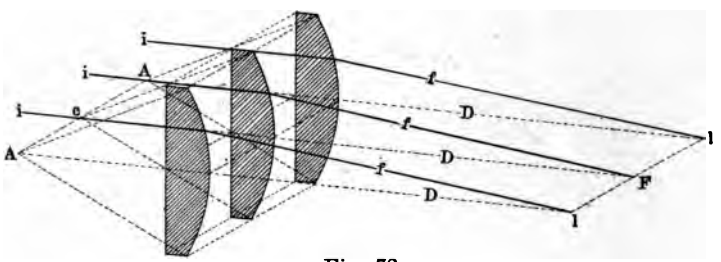

Fig. 72.

Moreover, the principal sections, Fig. 72, of the lens are parallel and have their corresponding focal points at the same distance, D, from the lens, thus producing a succession of adjacent points, resulting in the so-called focal line, lFl, which is parallel to and in the same plane with the axis, AcA, of the cylinder.

The diagrams, figures 73, 74, 75 and 76, are presented in order to more forcefully impress upon the mind the property of refraction with respect to the principal meridians of cylindric lenses. In fact, dexterity in the practice of adapting lenses to vision largely depends upon a clear conception of the principal planes of refraction, both in the human eye and in the lenses used to correct its various anomalies. Therefore, elementary as the accompanying discussion may appear, it nevertheless has its value as a mental drill in the acquirement of such knowledge.

As simple cylindric lenses have their surfaces of greatest obliquity

in the plane which is perpendicular to the axis, we here also find the refraction active in this plane, and passive in the axial or right-angled coordinate plane, wherein, the same as in the illustration of prisms, i_0 and f_0 are associated with refraction in the horizontal, and i_1 and f_1 with refraction in the vertical plane.

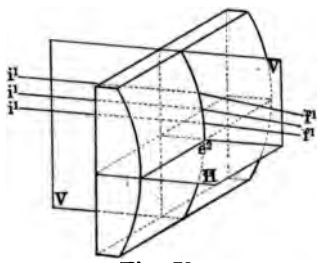

Fig. 73.
Axis, horizontal. Refraction, vertical.

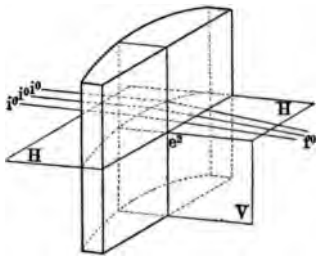

Fig. 74.
Axis, vertical. Refraction, horizontal.

Plano-convex Cylindric Lenses.

Furthermore, it is evident that the refraction in any plane that is placed diagonally between the passive and active planes of the cylinder will be equal to an amount varying between nil and its maximum refraction; and in case the cylinder is placed diagonally to a pair of right-angled coordinate planes, V and H, the sum of the dioptral

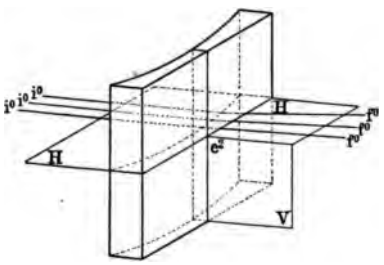

Fig. 75.
Axis, vertical. Refraction, horizontal.

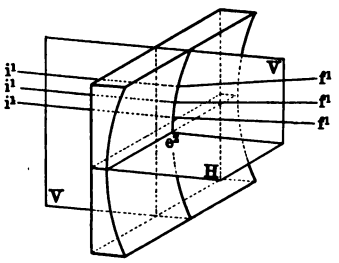

Fig. 76.
Axis, horizontal. Refraction, vertical.

Plano-concave Cylindric Lenses.

powers of the cylinder in these planes is always equal to the maximum dioptral power, D, of the cylinder. Thus, if β is the angle between the axis and the horizontal plane, H, the powers, D_V and D_H, in the planes, V and H, respectively, are related to each other as follows:

$$D_V \cos^2\beta + D_H \sin^2\beta = D, \quad \ldots\ldots\ldots\ldots\ldots \text{XXI.}$$

wherein D_V and D_H merely designate the same dioptral factor, D, in the planes, V and H.

In a practical experiment in which the lens is held at some distance from the normal eye, convex cylindric refraction manifests itself by an apparent *increase,* and concave cylindric refraction by an apparent *decrease* in the dimensions of an observed object in that plane which is at right angles to the axis. In the axial plane, the refraction being passive, corresponding dimensions remain unchanged.

To obtain cylindric refraction of equal amount in both planes, thereby reducing the focal line to a focal point, it is necessary to com-

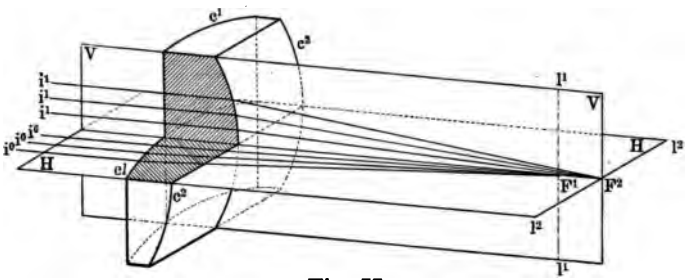

Fig. 77.
Double or Bicylindric Lens.

bine two identical cylinders, or, to create a single lens whose opposite surfaces are right-angled coordinate cylindric elements as shown in Fig. 77.

Under such circumstances, however, the focal line, $1_1F_11_1$, for the front surface, c_1, is slightly closer to the face of the lens than the focal line, $1_2F_21_2$, for the back surface, c_2. Aside from this, in making a

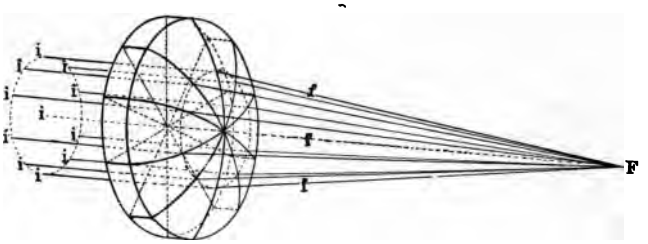

Fig. 78.
Plano-convex Spheric Lens.

bicylindric lens it is difficult to insure its chief planes of refraction being strictly at right-angles to each other, and which, if not the case, not only alters the maximum and minimum refractions, but also the positions of the principal planes in which they occur.

The greater the distance apart of the surfaces, c_1 and c_2, the greater will be the aberrative distance, F_1 to F_2; yet, as the thickness of the

lens may generally be accepted as a vanishing dimension in proportion to the focal distance, we may consider a common focal point to exist for both refracting surfaces.

However, it is far less difficult to create a *single* surface capable of producing this equal amount of refraction in both the vertical and horizontal planes.

In fact, such a surface is obtained through rotation upon the optic axis of the plano-convex section, 1, Fig. 6, whereby a plano-convex spheric lens, Fig. 78, is produced. Similar rotation of the remaining sections in Fig. 6 would produce the spheric lenses mentioned in connection therewith.

It is evident that the incident and final rays will retain their relative obliquity during the rotation, so that all incident parallel rays have their corresponding final rays in the resulting cone whose apex is at the focal point, F.

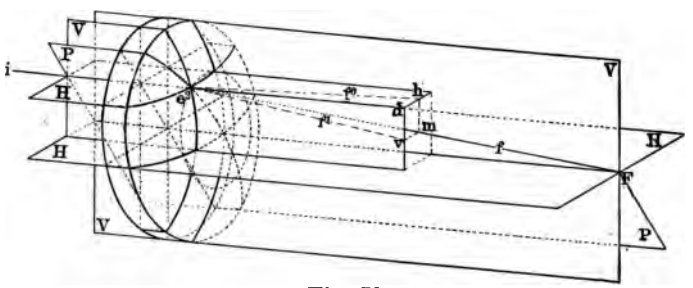

Fig. 79.

In order to further illustrate this, the principle previously utilized for the prism, Fig. 31, may also be applied to a lens having only one spheric surface, and which is consequently oblique with respect to both right-angled coordinate planes.

In the plano-convex spheric lens, Fig. 79, if we consider the refraction at e_2 of the ray, i, merely with regard to the horizontal refraction, the final ray would take the direction f_0 to h, and, if independently for the vertical refraction, the final ray would assume the direction f_1 to v. Therefore, with due consideration to the refraction in both planes, the refracted ray must include both properties of deflection, and result in a final ray, f, which is directed to the focal point, F, through a point, m, of the oblique plane, P, as defined by projection of the apportioned horizontal and vertical displacements, dh and dv.

Finally, we may, therefore, conclude that spheric refraction is equivalent to the refraction of right-angled crossed cylinders of identical curvatures.

As in spheric lenses the refraction is equally active in any pair of diametrically-opposed meridians, it follows that both the lateral and vertical dimensions of objects seen through them will appear to be enlarged by convex lenses and diminished by concave lenses, when these are held at some distance from the normal eye. In fact, persons who are expert have been known in this manner to accurately estimate the power of a lens by its associated apparent degree of magnification or minification of the distant object.

Moreover, on viewing a fixed distant object through a convex lens, held at about 12 inches from the normal eye, that is being shifted either vertically or horizontally across the visual line, the object will appear to move contrary to the motion of the lens; whereas, when a concave lens is used the motion of the object will agree with that of the lens. This principle is commonly utilized to ascertain the powers of ophthalmic lenses, through successively covering the lens, whose power it is desired to determine, by a contra-generic lens of known power, until the lens is found that neutralizes or arrests the apparent motion of the object. As a rule few lenses are required to be used before attaining neutralization, as the operator is generally able to at least closely approximate the power of the lens through having first applied the less accurate visual estimate.

Cylindric lenses are used to counteract astigmatism in the human eye, which, in other respects, may be either normal, farsighted or nearsighted. In the first mentioned instance a plano-cylindric lens is used to neutralize the astigmia; whereas, in hyperopia or myopia, the cylindric surface is necessarily combined with the required spheric surface, such combinations being called *sphero-cylindric lenses*. When a prism is combined with a cylinder, it constitutes a *cylindro-prismatic lens*, and in case a spheric surface is also incorporated, it is called a *sphero-cylindro-prismatic lens*; these terms being used to distinguish them among the great variety of *compound lenses* in general use to counteract astigmatism of the eye.

Astigmatism.

In *optics*, astigmatism is the failure of rays that are refracted or reflected at a surface to converge to a single point, and occurs when the *axis* of the incident pencil is *oblique* to one or both surfaces of a prism, lens, or mirror. For instance, in the case of refraction of an oblique bundle of rays near the periphery of a spheric surface, Fig. 80, the meridian-rays, of which the chief ray, ie, is the axis, come to a focus sooner upon the refracted chief ray, ef, than the sagittal rays, which suffer refraction to a lesser degree in a plane, pe, at right angles

to the meridian-plane, ec, at the same point, e. The point, f_1, which is the vertex of the meridian rays, is called the *primary image-point;* and the point, f_2, which is the vertex of the sagittal rays in the plane at right-angles to the meridian-plane, is called the *secondary image-point.* The lines, $d_1 f_1 d_1$ and $d_2 f_2 d_2$, through both of which all the rays of the refracted bundle pass, and which may be regarded as straight lines in their respective coordinate planes, are, in general, called the two image-lines of the narrow refracted bundle. In other words, the primary image-line lies in the primary principal section, and the secondary image-line lies in the secondary principal section, both of which are at right-angles to each other and perpendicular to

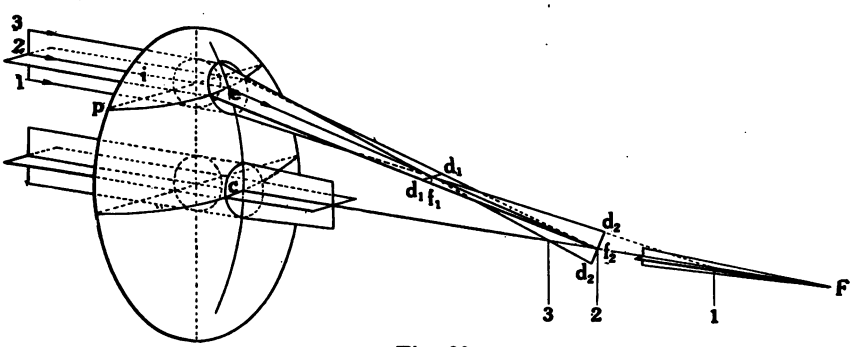

Fig. 80.
Astigmatic Refraction by a Plano-convex Lens.

the chief ray, ef_2, of the astigmatic bundle. This principle was established by Sturm, the originator of the theory of astigmatism. Matthiessen contends that the image-lines may not be, and, generally, will not be, perpendicular to the chief ray, ef_2, but it is more convenient and quite permissible to consider both of the image-lines, according to Sturm's definition, "perpendicular to the chief ray of the astigmatic bundle." In fact, they are so located for the cornea of the eye having regular astigmatism, and whose surface at the point of incidence, the vertex of the cornea, may be considered to be asymmetrical, owing to the difference of curvature in the principal sections or meridians exposed to *homocentric rays* (those that emanate from one and the same point on the axis) as well as *paraxial rays* (those that are incident within a very small area around the vertex of the surface). All other rays which do not lie in the principal sections will be refracted so as to intersect corresponding points within the limitations of the image-lines, as shown in Fig. 89A. In 1825, Airy was the first to apply a lens to correct the astigmatism in his own eye. As the refracted

bundle of rays thus formed does not anywhere pass through a point, it is called an astigmatic bundle or pencil. The same principle applies to a normally incident homocentric bundle of rays, provided the bundle is refracted or reflected by a toric surface; that is to say, a rotary surface in which the meridians of greatest and least curvature are at right-angles to each other. The toric surface, Fig. 98, is well illustrated through its comparison with the peripheral surface of an inflated rubber tire, which has a greater curvature along the circumference of the wheel than in the transverse plane of a spoke. In fact, the astigmatic cornea is such a surface. When incident paraxial and homocentric rays proceed from *infinity* and are refracted by a toric surface, or its equivalent, a sphero-cylindric lens, the image-points and image-lines are called *focal points* and *focal lines,* respectively. Hence, the primary and secondary focal points on the axis are in their corresponding *primary* and *secondary focal lines,* these being located in their correlated *primary* and *secondary focal planes,* which are perpendicular to the axis. The distance between the focal lines or planes at F_1 and F_2, Fig. 89A, is known as the "focal interval" of Sturm; and a third plane, situated between and parallel to the focal planes, contains a circle, defined by the outermost rays of the astigmatic bundle, which has been somewhat misleadingly called the circle of least confusion, although here perhaps is obtained the image-point having least distortion. The locus of this circle, if named the region of transition, would more definitely define its characteristic property and position, since it is that place between the focal planes, where the constantly varying elliptic cross sections of the refracted bundle, on the side of the approaching light-wave in front of the circle, are again converted into successive and similarly inconstant elliptic cross-sections whose diameters are reversed behind the circle, T, Fig. 89, which is, therefore, a characteristic cross-section of the astigmatic bundle. The theory of astigmatism for oblique incidence also applies to any spheroid surface, that is to say, to any rotary surface approaching a spheric form and that may be generated through rotation of one of the conic sections, such as the ellipse, the parabola or the hyperbola, and which are known as ellipsoids, paraboloids, and hyperboloids, respectively. For instance, a lens of the latter type is called a hyperbolic lens (rather difficult of production), and is occasionally used, with more or less success, to correct conical cornea, which has an acute curvature near its apex that flattens out and steeply approaches the limbus. This shape of the cornea suggested the possibility of correcting the defect by means of a hyperbolic lens, which also has this characteristic form. In practice, however, a difficulty is

encountered through the necessary movements of the eye in its orbit, causing the apex of the cornea to wander from the center of the lens, which should at all times be in collimation with the apex of the cornea, in order to make the lenticular correction efficient at the fovea centralis. For this reason hyperbolic lenses can only achieve their greatest efficiency when the apices of both the cornea and the lens are coincident with the line of fixation.

The astigmatism near the periphery of the conical cornea may, therefore, be said to be the dividing line between corrigible *regular* and *irregular astigmatism;* the latter comprising a variety of malformations of the corneal surface, or of the lens itself, that precludes satisfactory correction by lenses. Regular astigmatism is counteracted by means of cylindric or toric surfaces, such as are described under compound and toric lenses.

Corrigible astigmia of the eye implies that the visual line shall coincide with the optic axis of the corrective lens whose meridians of greatest and least refraction shall also correlatively coincide with the meridians of least and greatest refraction of the eye, in order to produce at its fovea an axial image-point instead of the diffusion produced by the refracted astigmatic bundle. Therefore, in ophthalmic lens-corrections only *direct pencils* of light are involved, namely those that are emitted from a luminous point upon the optic axis, and which, upon being refracted, are directed to a conjugate point upon the same axis.

It is nevertheless important that some consideration should also be given to the refraction of *oblique pencils,* on account of the aberration they so often produce when the visual axis of the eye and the axis of the corrective lens do *not* coincide. In fact, such distortion, detrimental to clear vision, may be produced through extreme excursions of the visual line from the center of any corrective lens, or by an abnormally tilted position of the lens itself before the eye. Incidentally, it may be mentioned that even moderately strong cylindric lenses, whose axes are *horizontal,* on being tilted before the visual line directed for distance, produce aberration; or, in case such lenses are properly placed for viewing remote objects, the downward direction of the visual line, incident to reading, produces the same effect. Therefore, when the axes of such cylindric lenses are *horizontal,* it is advisable to slightly decrease their power in case they are to be used for both reading and distance. Moreover, even a spheric lens that is tilted upon its optic axis produces an astigmatic pencil whose image-lines are closer to the lens than its true focus.

In fact, in any case where the axis of the incident cone of light does

not coincide with the normals to both opposite surfaces of the refracting medium, whether it be a lens, prism or plate, the refracted pencil will no longer be a circular cone of light; but, it will be a pencil bounded by a surface that penetrates and defines the illuminated area of the medium and two image-lines, which are at right angles to each other and the axis of the refracted pencil, Fig. 83.

The same law applies to obliquely incident pencils of light that are *reflected* by spheric surfaces. Its mathematical elucidation may be found in most works on geometric optics.

In illustration of the difference between refracted direct and oblique

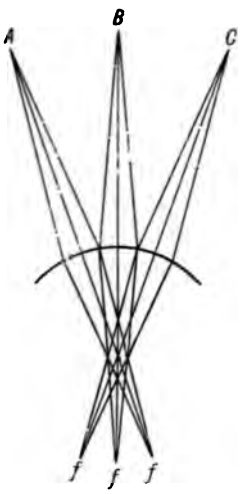

Fig. 81.

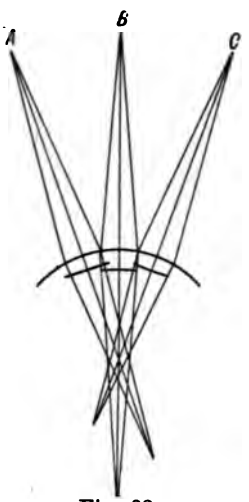

Fig. 82.

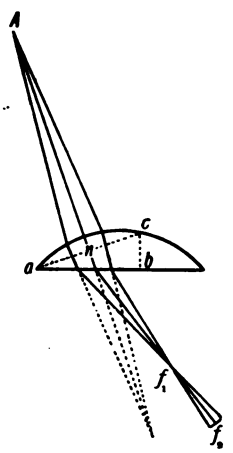

Fig. 83.

pencils, let the curved line in Fig. 81 represent the spheric surface of a medium whose density is greater than air, when perpendicularly incident conic pencils of light, projected upon it from successive points, A, B, C, will have their respective conjugate foci, f, upon the correlative radii with which the axes of the incident pencils coincide.

If the refracted pencils, within the medium, are to have *focal points outside* of the medium, the axes of these pencils will have to be *perpendicularly* intersected by the second surfaces, as shown by the heavy lines in Fig. 82; and in the event of the second surface occupying an oblique position, ab, Fig. 83, with respect to the pencil, A, the medium must be considered as a lens, having its optic center upon the axis, A n, of the incident pencil, with the prism, abc, added to it.

The circular cone of light, *within* the medium, will then project an elliptic area of illumination, E, Fig. 84, upon the second surface, as

the *axis* of the pencil is here *oblique,* so that the refracted pencil ceases to be a circular cone, projecting itself outside of the medium as an *astigmatic pencil,* of which f_1 and f_2 are the image-lines at right angles to the axis, which is deflected toward the base of the inherent prism, P.

This optical phenomenon, which is sphero-cylindro-prismatic in effect, has been known to physicists ever since Kummer, in 1860, first called attention to the theory by which it was mathematically proven.

It may be experimentally, though crudely, demonstrated by placing

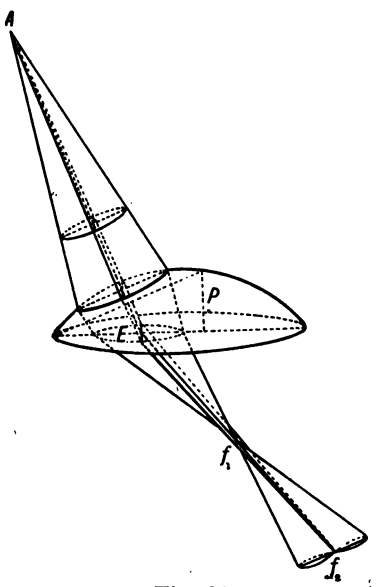

Fig. 84.

a plano-convex lens of 8 D. directly between a light at 20 feet, and a screen receiving its image. On interposing a prism of 20$^{\Delta}$, for example, with its base downward, and in a manner to insure contact of the plane faces of the glasses, the image will be observed to change both its form and position upon the screen. By drawing the screen slightly nearer to the lens, a horizontal though imperfectly defined line, corresponding to f_1, will become manifest, and by increasing the distance between lens and screen, a vertically elongated looped figure involving "coma-aberration," * closely resembling a line, at f_2, will appear.

When a circular cone of light, C, Fig. 85, from a short finite distance, falls obliquely upon the face of a prism, we again have an

* Some excellent drawings exhibiting these appearances are to be found in H. Dennis Taylor's *A System of Applied Optics* (London, 1906).

elliptic area of illumination, and the refracted rays, *within* the medium, will assume a direction as if emitted from the image-lines, v_1 and v_2, reaching the second surface of the prism, and being refracted by it to the eye at E, as if projected from the lines, V_1 and V_2, on the opposite side of the prism.

There is one exception to this result, and that is when the axis of the incident pencil assumes the direction which is subject to minimum

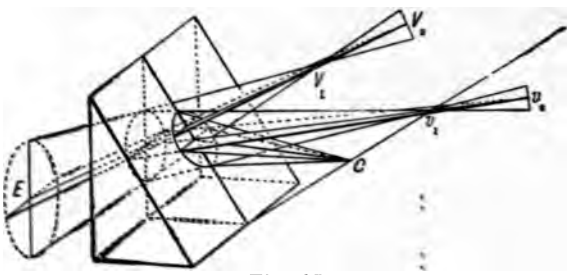

Fig. 85.

deviation, and in which event the emergent pencil will appear to diverge from a *point*, at the same distance from the anterior surface as the original source of light, C. In the case of a plate, the emergent rays will also constitute an astigmatic pencil, with the difference that its rays will appear to proceed from a pair of image-lines located upon an axis *parallel* to the axis of the incident pencil.

This sphero-cylindro-prismatic action, on the part of a simple prism,

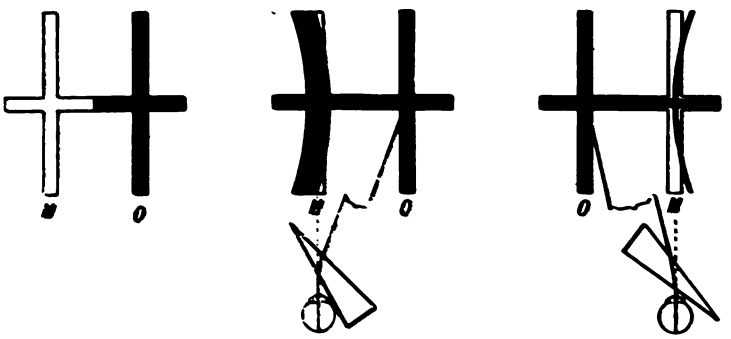

Fig. 86.

may be experimentally demonstrated in the following manner: Construct the figure MO (to the left in Fig. 86, in which the width of the principal bars is, say, 2 inches, and the distance apart of the perpendiculars is about 24 inches), and place it at right-angles to the line of sight, at a distance of about 6 feet from the eye, before which a prism

of 10$^{\Delta}$ is given considerable inclination to the visual axis, with its base in or out, as shown in the two right-hand diagrams, Fig. 86. The eye in each instance is to be placed directly opposite to the figure, M. In both cases the prism is shown not only to have changed the position of the solid cross, O, but also to have altered the dimensions of its vertical and horizontal bars in comparison with M.

With this knowledge of the refraction of oblique pencils, it is easy to comprehend the error committed in attempting to measure the true power of prisms by their apparent tangent-deflections observed at the distance of one meter, and why prisms, even of moderate power, produce aberration when used for reading at a lesser finite distance.

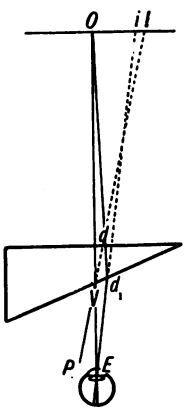

Fig. 87.

In Fig. 87 the relative positions of the object of fixation, O, at one meter's distance, the prism of exaggerated dimensions and the eye are shown. It is evident that the perpendicularly incident axis, OV, of the conic pencil of rays emitted by the object, O, coincides with the visual axis, and that the axis of the refracted pencil, VP, *does not* enter the eye, although it *does* define the deflection, OI, which it is desired to measure. However, the axis of the refracted pencil, d_1E, which *does* enter the eye, results from that incident pencil whose axis is *oblique* relatively to the normal at d, so that it is a ray approaching the direction for minimum deviation and must consequently suffer less deflection, Oi, than the refracted pencil whose axis is VP.

Therefore, it is evident that the observed deflection, Oi, is less than the actual deflection, OI, whose extent is really sought.

Even *thickness,* a dimension which we are taught to neglect with respect to ophthalmic lenses, becomes an appreciable and misleading factor in prisms above 8$^{\Delta}$, when the unjustifiable attempt is made to

measure their deflection at short finite distance. This will be apparent from the following considerations.

It has been shown that the ray, which in the nearest limit reaches the eye, is the axis, Od, Fig. 87, of an *oblique* pencil, being refracted within the prism, ABC, from d to d_1, and thence in air to the eye, E, which projects it to i, upon the scale, OI. For a given thickness of prism, this is the only pencil which will be received by the eye, since, if we increase the thickness, as in Fig. 88, by allowing the plane, A_1B_1, to represent the anterior surface of the prism, the original incident axis, Od, will be refracted at v instead of d, when the axis of the refracted pencil will traverse the path, vv_1P_1, to the left of the eye,

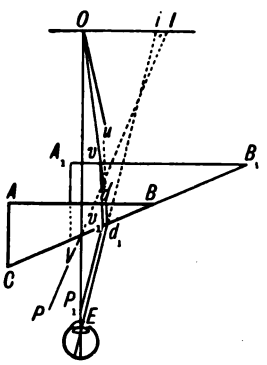

Fig. 88.

and parallel to dd_1E. The refracted pencil which would enter the eye, for the indicated *increased* thickness, could only accrue from an increased obliquity of the incident axis, Ou. The latter would, therefore, even more closely approach the direction for minimum deviation, and from which we are to conclude that the deflection noted upon the scale, OI, by the eye will be *least near the base* and consequently greatest near the apex of the prism.

In fact, this was proven by actual experiment with a prism 1¼ inches square, whose apical angle was 22°, and which, by an observation taken near the base, indicated 17.9$^{\Delta}$; whereas, through the feather-edged apex it measured 18.9$^{\Delta}$. The same prism placed 6 meters from the scale measured 20$^{\Delta}$. Therefore, the error committed by measurement through the apex was 1.1$^{\Delta}$, while the increased thickness at the base still further increased the error by 1$^{\Delta}$. Consequently, the conclusion is justified that *the error will be least in prisms of high degree,* when readings at finite distance are taken at the apex of the prism, and that it will be reduced to a minimum, *throughout the principal refract-*

ing plane, when the deflection is measured for pencils which are *perpendicularly incident to all points of the prism-surface,* that is to say, when the pencils of light are *cylindric,* and which will practically be the case when the object of fixation, *a line, is situated at 6 meters' distance.* Thus, it has been conclusively demonstrated that, the power of a prism can not be accurately determined, through observation of the apparent tangent displacement of an object seen through it, at any shorter distance than 6 meters.

Compound Lenses.

I. Congeneric Meridians (Convex).

Among ophthalmic lenses an astigmatic lens is one in which the diametrically opposed principal sections include different degrees of refraction, in contradistinction to those hitherto mentioned, in which

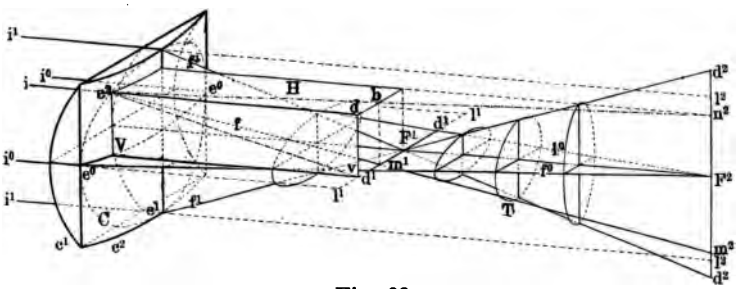

Fig. 89.

Convex Bicylindric Lens ($+ c_1$, axis 180° $\supset + c_2$, axis 90°).

uniform refraction took place either exclusively in one meridian, or equally in both principal meridians.

Referring to Fig. 77, it is evident that the aberrative distance, F_1 to F_2, may also be definitely increased through assigning different amounts of refraction to the active planes or sections of the combined cylinders. In this event the focal point ascribed to the equally curved bicylindric lens will be effaced, though substituted by a pair of focal lines whose distance apart will be equal to the difference between the focal distances of the crossed *unequal* cylinders. Thus, in the bicylindric lens, Fig. 89, consisting of two crossed convex cylinders, c_1 and c_2, of unequal curvatures, $l_1F_1l_1$ and $l_2F_2l_2$ will be the corresponding *elementary* focal lines. The distance between them, F_1 to F_2, is the "focal interval" of Sturm.

As the cylinders are of equal length, the focal lines, $l_1F_1l_1$ and $l_2F_2l_2$, would also be identical in this respect, if the apportioned refractions of the cylinders were considered *independently* of each other.

The combined refraction of the cylinders, however, definitely modifies this hypothesis, and in the following manner:

The outermost incident rays, i_o, in the central horizontal plane, which would have been directed to the points l_1 and l_1, for the cylinder c_1, will suffer horizontal displacement toward the point, F_2, owing to the activity of the refraction in this plane for the cylinder, c_2, and so establish points d_1 and d_1, of the focal line, $l_1F_1l_1$, for the combined action of the cylinders, c_1 and c_2, in the *horizontal* plane.

Similarly, the outermost incident rays, i_1, in the central vertical plane, which would have been directed to the points l_2 and l_2, for the cylinder, c_2, will suffer vertical refraction in this plane by the cylinder, c_1, which causes the final rays to cross each other at F_1 and to intersect the focal line, $l_2F_2l_2$, at the points d_2 and d_2, for the combined action of the cylinders, c_1 and c_2, in the *vertical* plane.

If we consider the refraction at the point e_2 of the circle, C, for the ray, i, merely with regard to the horizontal refraction of the surfaces, or the cylinder, c_2, the final ray would take the direction e_2h, intersecting the focal line of the cylinder, c_2, at a correlative point, n_2; but, as all final rays for the cylinder, c_1, above the central horizontal plane, intersect the focal line, $d_1F_1d_1$, it follows, through presence of the cylinder, c_1, that the ray, e_2h, must fall subject to the influence of c_1 for the combined action of the cylinders, thus depressing the ray, e_2h, from the point h, perpendicularly to m_1, and consequently also the point n_2 to m_2, within the focal line, $d_2F_2d_2$.

Therefore, by means of the same reasoning that was applied to the refraction of a rotated prism, Fig. 31, it is found that the direction of the final ray, f, has, in this instance, also been determined by projection of the apportioned horizontal and vertical displacements, dh and dv, which are solely dependent upon the active meridians of the cylinders, c_1 and c_2.

Increased proximity of the point e_2 to e_o, upon the circle, C, will be associated with an increased distance between m_1 and F_1, and with an approach of m_2 toward F_2 for these points of intersection of the final ray, f, within the respective focal lines, F_1d_1 and F_2d_2. The reverse is evident for an advancement of e_2 toward e_1.

The total refraction for all incident parallel rays within the area of the circle, C, Fig. 89A, will, therefore, result in an astigmatic pencil whose focal lines, $d_1F_1d_1$ and $d_2F_2d_2$, are limited as to position and magnitude. This astigmatic pencil, if intercepted at intervals by a transverse perpendicular screen, will thereon project elliptic areas of light whose longest and shortest diameters correspond to the principal meridians of refraction. In the immediate vicinity of F_1, for instance,

the ellipses have their longest diameters horizontal; whereas, in the vicinity of F_2 their longest diameters are vertical.

This naturally effects a reversal of the ellipses, respecting their diameters, at some point within the focal interval, F_1F_2; such point being determined where the vertical and horizontal displacements are alike, and the section, T, Fig. 89, is a circle whose locus is the region of transition.

Astigmatic refraction in a lens is, however, preferably attained by combining a spheric with a cylindric surface; the requisite conditions being fulfilled through that increase or decrease of the spheric refraction which is produced by and in the active meridian of the cylinder. To increase the refraction of a positive or negative spheric lens in

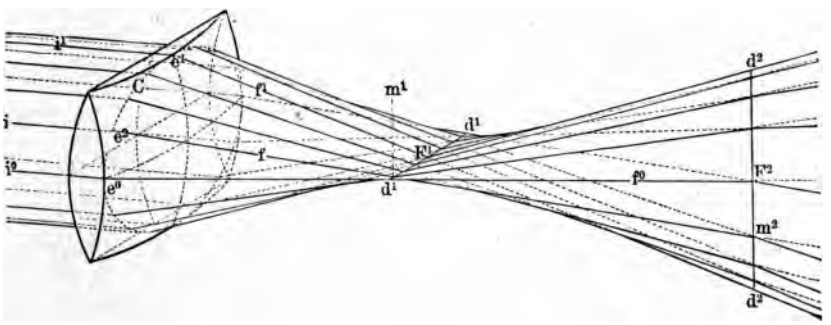

Fig. 89A.

one meridian, we may add to it the active meridian of the cylinder bearing the same sign; and to decrease it in the same meridian we may combine it with the active meridian of a cylinder bearing the opposite sign. For instance:

1. The combination of a positive spheric with a positive cylindric surface results in the section of *greatest* refraction being double convex; and,

2. The combination of a positive spheric with a weaker or less acutely curved negative cylindric surface results in the section of *least* refraction being *periscopic* convex.

Where the aforesaid combinations are spoken of, we shall, for convenience, apply to them the terms double and periscopic form, respectively.

As the combination of crossed convex cylinders of unequal curvatures gave rise to a pair of focal lines, to the novice it may appear requisite that a focal point and a focal line should exist for the com-

bination of a spheric with a cylindric surface. However, the following analysis will avert this possible though erroneous impression:

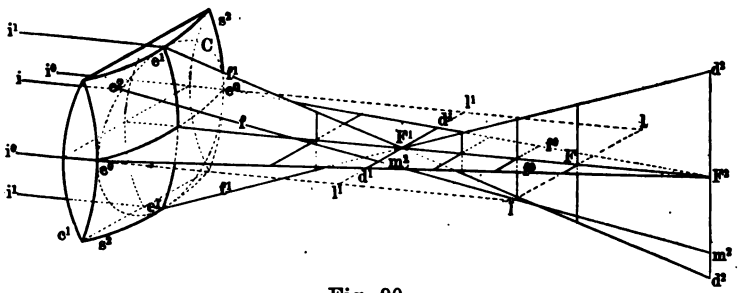

Fig. 90.

Convex Sphero-cylindric Lens ($+ s_2 \supset + c_1$, axis 180°).—Double Form.

In the convex sphero-cylindric lens of double form, Fig. 90, if we considered the refraction for each surface independently of the other, we should find a focal point at F_2 for the convex spheric surface, s_2, and a focal line, say at l F l, for the cylindric surface, c_1. Their combination giving rise to augmented refraction in the vertical plane, however, occasions a displacement of the focal line, lFl, to the position of $l_1F_1l_1$.

The final rays from the outermost points, e_0, in the horizontal plane, being directed to the focal point, F_2, it is evident that the focal line, $l_1F_1l_1$, must become subject to the influence of the spheric refraction in

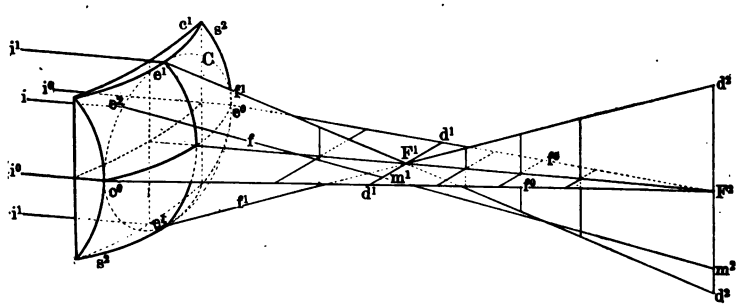

Fig. 91.

Convex Sphero-cylindric Lens ($+ s_2 \supset - c_1$, axis 90°).—Periscopic Form.

this plane, thereby establishing the points d_1 and d_1, and restricting the magnitude of the focal line to $d_1F_1d_1$.

The final rays from the outermost points, e_1, in the vertical plane, which in absence of the cylinder would have been directed to the focal point, F_2, now cross each other at F_1. Their extremities are, therefore, displaced from F_2 to d_2 and d_2, thus resulting in the destruction of the

focal point, F_2, and establishing a limitation of the rays to a *created* focal line, $d_2F_2d_2$.

The convex sphero-cylindric lens of periscopic form, Fig. 91, is constructed by combining a *weaker* concave cylinder, c_1, with a convex spheric surface, s_2; the axis of the cylinder here being placed in the vertical instead of the horizontal plane for the purpose of future comparison.

In this case we have given to the spheric surface, s_2, a curvature corresponding to the focal point, F_1, and to the cylindric surface, c_1, a curvature which, acting in combination with its associated horizontal meridian of the spheric surface, causes the rays to unite at the focal line, $d_2F_2d_2$. The reason given for the destruction of the focal point F_2, in the lens, Fig. 90, may in this instance be similarly applied to explain the creation of the primary focal line, $d_1F_1d_1$, as well as the limitation of the secondary focal line to the magnitude of $d_2F_2d_2$.

A characteristic difference between the double and the periscopic form of astigmatic lens consists in the fact that the positions of the focal lines are interchanged with respect to their correlative elements of creation. Thus, in Fig. 90 the focal line, $d_2F_2d_2$, corresponds to the initial effect of the spheric surface; whereas, in Fig. 91 the primary focal line, $d_1F_1d_1$, corresponds to it.

This difference, however, is not material, as it is evident that the magnitude of the focal lines and their distances from the lens are dependent upon the refraction ascribed to its two principal sections; and, since any two given points (d_1 and F_2, F_1 and d_2, m_1 and m_2) definitely fix the position of a line or ray in space, it is further obvious that the direction of all final rays will be identical for any thin lens in which the right-angled coordinate meridians of greatest and least refraction are allotted the same, and upon whose surface the rays are incident in the immediate vicinity of the optic axis.

To demonstrate the analysis of formulæ for these equivalents, we shall, in the respective figures, designate the refraction as being expressed by:

Ia.	+ 3.5 D. cyl., axis 180°	⌣	+ 1.5 D. cyl., axis 90°.	(Fig. 89.)
IIa.	+ 1.5 D. spheric	⌣	+ 2.0 D. cyl., axis 180°.	(Fig. 90.)
IIIa.	+ 3.5 D. spheric	⌣	− 2.0 D. cyl., axis 90°.	(Fig. 91.)

It being necessary to become thoroughly familiar with the meridians of greatest and least refraction, it is considered expedient to picture these in their respective planes of activity, V and H, as shown in their

correlative sectional diagrams, Fig. 89a, Fig. 90a, Fig. 91a, and to refer to them as follows:

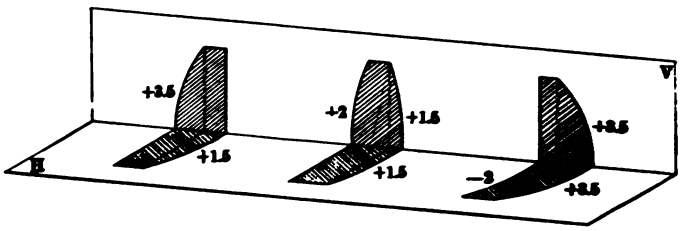

Fig. 89a. Fig. 90a. Fig 91a.

Formula Ia. + 3.5 D. cyl., ax. 180° ⌒ +1.5 D. cyl., ax. 90°. (Fig. 89.)
Refraction: + 3.5 D., vertical ⌒ + 1.5 D., horizontal, = + 3.5 V ⌒ + 1.5 H. Fig. 89a.

Formula IIa. + 1.5 D. spheric ⌒ + 2 D. cyl., axis 180°. (Fig. 90.)
Refraction: + 2 D. + 1.5 D., vertical ⌒ + 1.5 D., horizontal, = + 3.5V ⌒ + 1.5 H. Fig. 90a.

Formula IIIa. + 3.5 D. spheric ⌒ − 2 D. cyl., axis 90°. (Fig. 91.)
Refraction: + 3.5 D., vertical ⌒ − 2 D. + 3.5 D., horizontal, = + 3.5V ⌒ + 1.5 H. Fig. 91a.

Therefore, the lenses Ia, IIa, and IIIa are asymmetrically-refracting equivalents.

As the preference is generally given to the double form (Formula IIa), and, under certain circumstances, occasionally to the periscopic form (Formula IIIa), the rules applicable to the conversion of the one into the other formula alone are given.

To convert the double into the periscopic form:

Rule 1. *Place the sum of both numerals of refraction as the numeral for the newly created spheric element,* and combine with the same cylindric element having its sign and axis reversed.*

To convert the periscopic into the double form:

Rule 2. *Place the difference of both numerals of refraction as the numeral for the newly created spheric element,* and combine with the same cylindric element having its sign and axis reversed.*

As these lenses are practically only used for the correction of anomalies of ocular refraction, it is customary, when adapting them, to note the positions of the cylindric axes by means of the graduations

* The sign of the original spheric surface remaining unchanged.

upon the trial-frame. This, however, does not change the inherent properties of the lenses, whose meridians of greatest and least refraction are always 90° apart for all possible axial positions between 0° and 180°.

Thus, in the instance of the formula:

$$+1.5 \text{ D. sph.} \subset + 0.50 \text{ D. cyl., axis } 130°,$$

the periscopic form would be expressed, according to Rule 1, by

$$+2 \text{ D. sph.} \subset - 0.50 \text{ D. cyl., axis } 40°.$$

Inversely, the former may be made the result of the latter through application of Rule 2.

2. *Congeneric Meridians (Concave).*

The preceding analytic treatment is alike applicable to the similarly planned † concave compound lenses, Figs. 92, 93, 94, in each of which the focal lines, and consequently also the focal interval and region of transition are virtual and in the negative region before the lens.

All parallel rays incident upon and within the periphery of the circle, C, in any of the figures, will, therefore, result in final rays, behind the lens, which appear to emanate from correlatively established virtual points (d_1 and F_2, F_1 and d_2, m_1 and m_2) of and within the limits of the focal lines in front of the lens. For these lenses, respectively, the refraction has been allotted as follows:

Ib. — 1.5 D. cyl., axis 180° ⊂ — 3.5 D. cyl., axis 90°. (Fig. 92.)
IIb. — 1.5 D. spheric ⊂ — 2 D. cyl., axis 90°. (Fig. 93.)
IIIb. — 3.5 D. spheric ⊂ + 2 D. cyl., axis 180°. (Fig. 94.)

and which, through similar use of sectional planes, such as applied to Fig. 89a, etc., will be found to be asymmetrically-refracting equivalents.

According to Rule 1, for instance, the concave sphero-cylindric lens,

$$-1.25 \text{ D. sph.} \subset - 0.75 \text{ D. cyl., axis } 160°,$$

may be converted into the periscopic form:

$$-2 \text{ D. sph.} \subset + 0.75 \text{ D. cyl., axis } 70°,$$

and *vice versa,* according to Rule 2.

† The meridian of greatest refraction is here placed in the horizontal instead of the vertical plane.

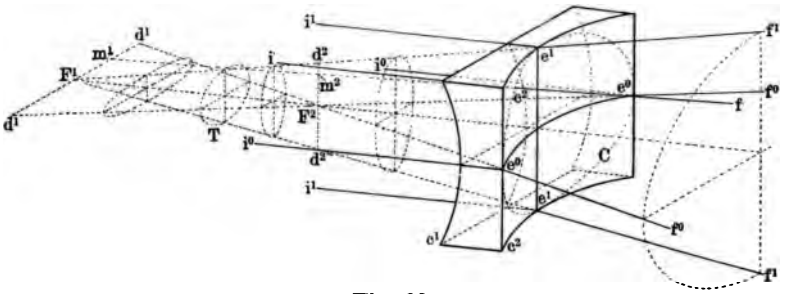

Fig. 92.

Concave Cylindro-cylindric Lens ($-c_1$, axis 180° $\supset -c_2$, axis 90°).

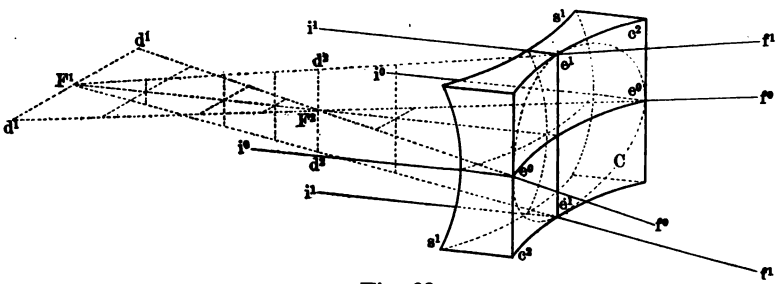

Fig. 93.

Concave Sphero-cylindric Lens ($-s_1 \supset -c_2$, axis 90°).—Double Form.

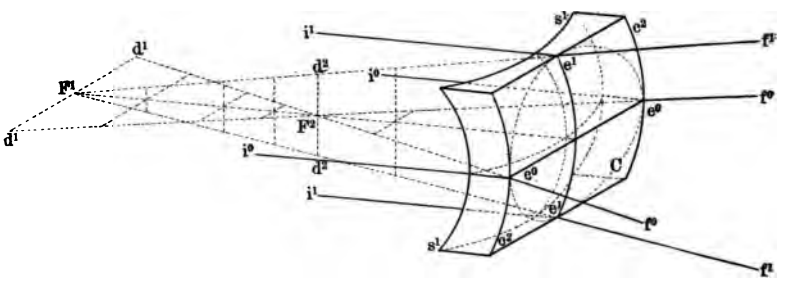

Fig. 94.

Concave Sphero-cylindric Lens ($-s_1 \supset +c_2$, axis 180°).—Periscopic Form.
In the above figures, i_0, e_0 and f_0 are associated with horizontal refraction, and i_1, e_1 and f_1 with vertical refraction.

3. *Contra-generic Meridians (Convex and Concave).*

As a final complication, different or even like degrees of positive and negative refraction may be ascribed to the diametrically opposed principal sections of a lens; this being achieved in the lens, Fig. 95, through combining a concave cylinder, c_1, with a convex cylinder, c_2, so that their contra-generic active meridians are at right-angles to each other.

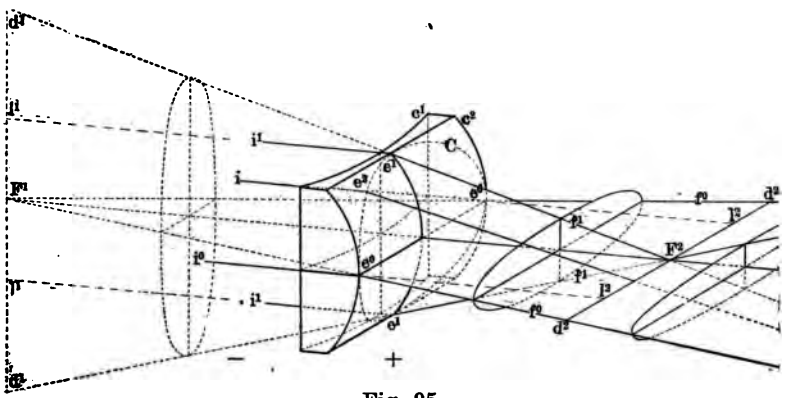

Fig. 95.

Concavo-convex Cylindro-cylindric Lens ($-c_1$, axis 90° $\supset + c_2$, axis 180°).

Independently considered, each cylinder, c_1 and c_2, would have its focal line, $l_1F_1l_1$ and $l_2F_2l_2$, of original magnitude in the region of its sign, $-$ and $+$, respectively, and consequently on opposite sides of the lens.

When the cylinders are associated, however, the final rays, which would have been restricted to the limits of the focal line, $l_2F_2l_2$, for the cylinder, c_2, will, by virtue of the dispersive effect of the cylinder, c_1, in the horizontal plane, be confined to an augmented focal line, $d_2F_2d_2$, within the limits d_2 and d_2, for the outermost rays emanating from the point, F_1, of the virtual focal line, $l_1F_1l_1$.

Therefore, all final rays within the limits of the circle, C, will be accorded associated vertical and horizontal refraction, culminating in their united intersection of a line, $d_2F_2d_2$, in the horizontal plane and in the positive region behind the lens. Interception of these rays, by successive transverse vertical planes, will make manifest similarly arranged ellipses respecting their greatest and least diameters, before and behind the focal line, $d_2F_2d_2$. By projecting the final rays into the region of their apparent emanation from before the lens, the virtual focal line, $l_1F_1l_1$, is increased in magnitude to $d_1F_1d_1$, so that the so-defined ellipses are reversed respecting their greatest and least

diameters, as shown by the dotted lines in the negative region, Fig. 95. Here the circle of transition is within the plane of the lens itself.

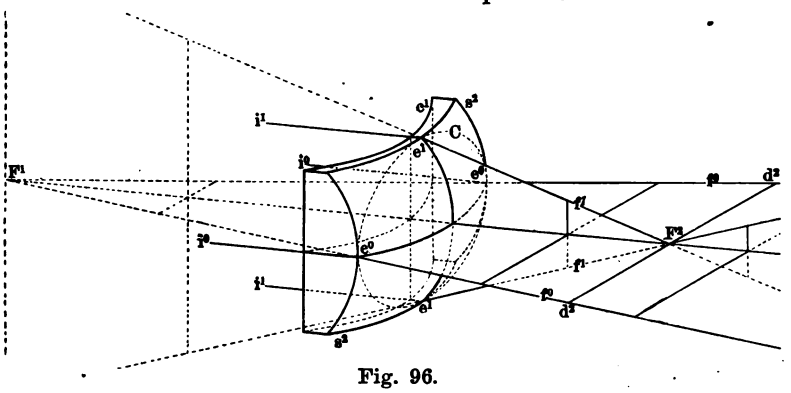

Fig. 96.

Concavo-convex Sphero-cylindric Lens ($+ s_2 \supset - c_1$, axis 90°).

Identical refraction is also preferably obtained in this instance by combinations of spheric with cylindric surfaces.

The combination of a convex spheric surface with the active meridian of a stronger concave cylinder creates a periscopic section which is concave; whereas, the combination of a concave spheric surface with

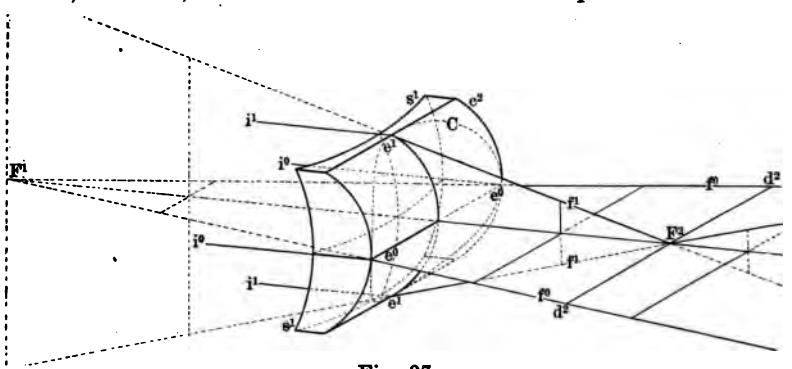

Fig. 97.

Concavo-convex Sphero-cylindric Lens ($- s_1 \supset + c_2$, axis 180°).

the active meridian of a stronger convex cylinder results in a periscopic section which is convex. The identity of the refraction for these combinations becomes apparent by reference to the concavo-
ıvex sphero-cylindric lenses, Figs. 96 and 97, in which, through a
ıs selection of the respective spheric and cylindric curvatures,
positive and negative elements of refraction for the prin-
f the crossed cylindric lens, Fig. 95, are fulfilled.
ı equality of formulæ characterizing these equiva-

lents, we refer to their correlative sectional diagrams, Figs. 95c, 96c, 97c, in the following order:

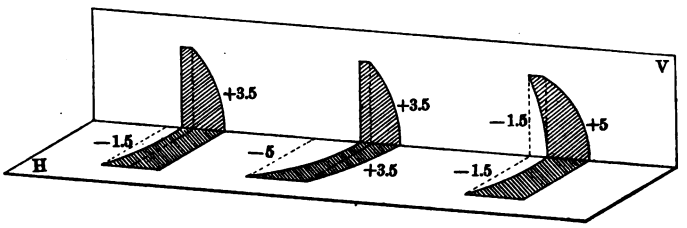

Fig. 95c. Fig. 96c. Fig. 97c.

Formula Ic. — 1.5 D. cyl., ax. 90° ⊂ + 3.5 D. cyl., ax. 180°. (Fig. 95.)
Refraction: — 1.5 D., horizontal ⊂ + 3.5 D., vertical = — 1.5 H ⊂ + 3.5 V. Fig. 95c.

Formula IIc. + 3.5 D. spheric ⊂ — 5 D. cyl., axis 90°. (Fig. 96.)
Refraction: — 5 D. + 3.5 D., horizontal ⊂ + 3.5 D., vertical = — 1.5 H ⊂ + 3.5 V. Fig. 96c.

Formula IIIc. — 1.5 D. spheric ⊂ + 5 D. cyl., axis 180°. (Fig. 97.)
Refraction: — 1.5 D., horizontal ⊂ — 1.5 D. + 5 D., vertical = — 1.5 H ⊂ + 3.5 V. Fig. 97c.

These lenses being equivalents, the rule is given only for converting the cylindro-cylindric lens (Formula Ic) into the concavo-convex sphero-cylindric lenses (Formula IIc and IIIc).

Rule 3. *Place the sum of both numerals of refraction as the numeral of the newly-created cylindric element, giving to it both the sign and axis of either cylinder, and combine with the neglected cylindric numeral and its associated sign as the spheric element.*

Comparison of the periscopic lenses, Figs. 91 and 94 with the Figs. 96 and 97, respectively, exhibits a striking similarity in their construction. The characteristic difference between them is that in the latter the cylindric refraction exceeds the spheric refraction, whereas, in the former the reverse is the case.

In the first edition of the author's work, *Ophthalmic Lenses,* 1886, tables were published which showed the combinations available, between 0.25 and 3.25 dioptries, for crossed cylindric lenses, together with their sphero-cylindric equivalents. In the accompanying abbreviated tabulation, Table I, the diagonal line of spheric lenses divides the table into two sets of compound lenses that are duplicates in refraction, even though there is a difference in the positions of the cylindric axes. Thus, all lenses in the vertical columns *beneath*

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the spheric lenses are correlative duplicates of the lenses in the horizontal columns to the *right* of them. For instance, $A_1 = a_1$, $A_2 = a_2$; $B_1 = b_1$, $B_2 = b_2$, etc.

I. TABLE OF CROSSED CYLINDERS AND THEIR SPHERO-CYLINDRIC EQUIVALENTS.

Congeneric Meridians (Convex).

DIOPTRIES.	+0.25 C. 180°	+0.50 C. 180°	+0.75 C. 180°	+1.00 C. 180°	+1.25 C. 180°
+0.25 C. 90°	(+0.25)	a^1 +0.25⊂+0.25⊖ +0.50⊂−0.25⊕	a^2 +0.25⊂+0.50⊖ +0.75⊂−0.50⊕	a^3 +0.25⊂+0.75⊖ +1.00⊂−0.75⊕	+0.25⊂+1.00⊖ +1.25⊂−1.00⊕
+0.50 C. 90°	A^1 +0.25⊂+0.25⊕ +0.50⊂−0.25⊖	(+0.50)	b^1 +0.50⊂+0.25⊖ +0.75⊂−0.25⊕	b^2 +0.50⊂+0.50⊖ +1.00⊂−0.50⊕	b^3 +0.50⊂+0.75⊖ +1.25⊂−0.75⊕
+0.75 C. 90°	A^2 +0.25⊂+0.50⊕ +0.75⊂−0.50⊖	B^1 +0.50⊂+0.25⊕ +0.75⊂−0.25⊖	(+0.75)	+0.75⊂+0.25⊖ +1.00⊂−0.25⊕	+0.75⊂+0.50⊖ +1.25⊂−0.50⊕
+1.00 C. 90°	A^3 +0.25⊂+0.75⊕ +1.00⊂−0.75⊖	B^2 +0.50⊂+0.50⊕ +1.00⊂−0.50⊖	+0.75⊂+0.25⊕ +1.00⊂−0.25⊖	(+1.00)	+1.00⊂+0.25⊖ +1.25⊂−0.25⊕
+1.25 C. 90°	+0.25⊂+1.00⊕ +1.25⊂−1.00⊖	B^3 +0.50⊂+0.75⊕ +1.25⊂−0.75⊖	+0.75⊂+0.50⊕ +1.25⊂−0.50⊖	+1.00⊂+0.25⊕ +1.25⊂−0.25⊖	(+1.25)

In the above formulæ the first numerals apply to spheric refraction, and the second numerals to the cylindric refraction. In the associated signs for axial position, the upright and horizontal diameters (| and —) of the circles denote the axes, 90° and 180°, respectively. With the exception of the diagonal line of spheric equivalents, each field contains both the double and periscopic forms of the convex sphero-cylindric equivalent. For crossed concave cylinders it is merely necessary to reverse the signs, + and —, wherever they occur.

II. TABLE OF CROSSED CYLINDERS AND THEIR SPHERO-CYLINDRIC EQUIVALENTS.

Contra-generic Meridians (Convex and Concave).

DIOPTRIES	+0.25 C. 180°	+0.50 C. 180°	+0.75 C. 180°	+1.00 C. 180°	+1.25 C. 180°
−0.25 C. 90°	+0.25⊂−0.50⊕ −0.25⊂+0.50⊖	+0.50⊂−0.75⊕ −0.25⊂+0.75⊖	+0.75⊂−1.00⊕ −0.25⊂+1.00⊖	+1.00⊂−1.25⊕ −0.25⊂+1.25⊖	+1.25⊂−1.50⊕ −0.25⊂+1.50⊖
−0.50 C. 90°	+0.25⊂−0.75⊕ −0.50⊂+0.75⊖	+0.50⊂−1.00⊕ −0.50⊂+1.00⊖	+0.75⊂−1.25⊕ −0.50⊂+1.25⊖	+1.00⊂−1.50⊕ −0.50⊂+1.50⊖	+1.25⊂−1.75⊕ −0.50⊂+1.75⊖
−0.75 C. 90°	+0.25⊂−1.00⊕ −0.75⊂+1.00⊖	+0.50⊂−1.25⊕ −0.75⊂+1.25⊖	+0.75⊂−1.50⊕ −0.75⊂+1.50⊖	+1.00⊂−1.75⊕ −0.75⊂+1.75⊖	+1.25⊂−2.00⊕ −0.75⊂+2.00⊖
−1.00 C. 90°	+0.25⊂−1.25⊕ −1.00⊂+1.25⊖	+0.50⊂−1.50⊕ −1.00⊂+1.50⊖	+0.75⊂−1.75⊕ −1.00⊂+1.75⊖	+1.00⊂−2.00⊕ −1.00⊂+2.00⊖	+1.25⊂−2.25⊕ −1.00⊂+2.25⊖
−1.25 C. 90°	+0.25⊂−1.50⊕ −1.25⊂+1.50⊖	+0.50⊂−1.75⊕ −1.25⊂+1.75⊖	+0.75⊂−2.00⊕ −1.25⊂+2.00⊖	+1.00⊂−2.25⊕ −1.25⊂+2.25⊖	+1.25⊂−2.50⊕ −1.25⊂+2.50⊖

In the above formulæ the numerals and signs have the same significance as in the preceding table, but for crossed cylinders having the axis of the concave cylinder at 180° and the axis of the convex cylinder at 90°, it is necessary to *reverse the axes* throughout.

Toric Lenses.

I. *Congeneric Meridians.*

The properties of astigmatic refraction are also fulfilled in a lens by creating for it, opposite to its plane side, a single surface whose diametrically-opposed principal meridians are of unequal refraction.

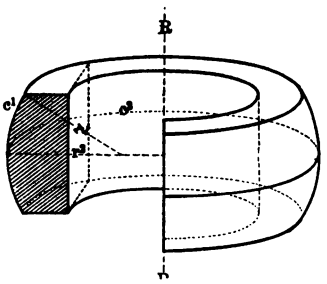


Fig. 98.

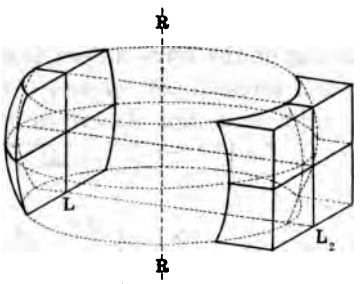


Fig. 99.

Such a surface, called a *torus,* is shown in Fig. 98, wherein the curvature, c_1, whose radius is r_1 and refraction is 3 D., is rotated upon a vertical axis, R, so as to create the curvature, c_2, whose radius, r_2, is chosen to produce 2 D.

In Fig. 99 two lenses are shown to be included within the surface so developed and an opposite plane side, the one being a plano-convex toric lens, L_1, the other a plano-concave toric lens, L_2.

From the construction it follows that these lenses are each possessed of 3 D. of refraction in the vertical meridian, and 2 D. of refraction in the horizontal meridian, so that the formulæ for the same may be expressed by:

(A_1) [+ 3 D., Ref. 90° ⌣ + 2 D., Ref. 180°] Tor.....(L_1)
(B_1) [− 3 D., Ref. 90° ⌣ − 2 D., Ref. 180°] Tor.....(L_2)

as a distinction from the correlative formulæ, A_2 and B_2, for a pair of crossed cylinders of identical refraction:

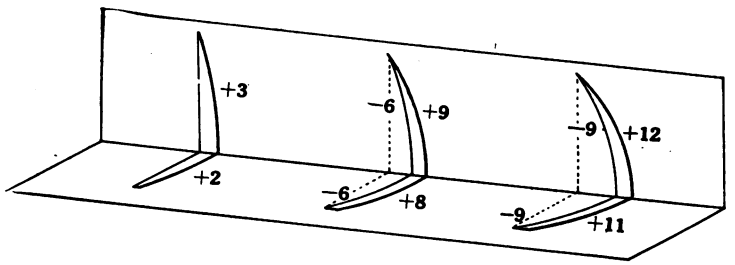
(A_2) + 3 D. cyl., axis 180° ⌣ + 2 D. cyl., axis 90°.
(B_2) − 3 D. cyl., axis 180° ⌣ − 2 D. cyl., axis 90°.

and their corresponding equivalents:

(A_3) { + 2 D. sph. ⌣ + 1 D. cyl., axis 180° (Double form).
{ + 3 D. sph. ⌣ − 1 D. cyl., axis 90° (Periscopic form).

(B_3) { − 2 D. sph. ⌣ − 1 D. cyl., axis 180° (Double form).
{ − 3 D. sph. ⌣ + 1 D. cyl., axis 90° (Periscopic form).

The plano-toric lens, L_1 or L_2, does not generally possess any advantage over its corresponding sphero-cylindric equivalent, so that the latter combination is preferably used. However, as the convex toric surface is especially well adapted to lenses of the deep meniscus and contra-meniscus types, it is usually combined with a concave spheric surface of more or less variable curvature, due to the fact that American manufacturers, in this instance, call the weaker curvature of the toric surface the "base-curve" of the sphero-toric lens. In this respect, the meaning of the term differs from that applied by them to the spheric menisci, whose inner concave surfaces are uniformly 6 dioptries. With this understanding American manufacturers require that all orders



Plano-toric. Toric Meniscus. Base-curve, 8. Toric Meniscus. Base-curve, 11.

Fig. 100.
Congeneric Meridians.

for toric lenses shall be written the same as for sphero-cylinders, except that the desired base-curve of the toric surface must also be specified; otherwise, toric surfaces with the 6-dioptry curve are usually supplied. In case this choice should result in the lens being comparatively shallow in depth, it is advisable to select a 9-dioptry curvature for the toric surface. Incidentally, a concave toric surface is rarely used in combination with a convex spheric surface.

The object of the toric surface is to provide a definite focal interval that may be utilized in combination with any spheric surface, in order to produce the same interval between the resultant meridians of greatest and least refraction of the so-called *sphero-toric lens.*

This is made clear through comparison of the principal sections in different lenses of the same power, pictured in Fig. 100, and in which the convex *toric* surfaces are drawn with *heavier* lines, in order to distinguish them from their associated concave spheric surfaces.

Each of these lenses has + 3 D. of refraction in the vertical plane and + 2 D. of refraction in the horizontal plane, and therefore corresponds to the sphero-cylindric lens, + 2 D. sph. ⌒ + 1 D. cyl., axis 180°.

The toric meniscus or contrameniscus usually affords the advantage of allowing its peripheral area to be brought nearer to and more concentric with the eye-ball than is possible with the sphero-cylindric equivalent; so that, for all ordinary movements of the eye-ball the visual axis will be less oblique to the inner surface of the lens. This and the consequent absence of reflection from the inner concave surface also give this form of sphero-toric lens a wider and more comfortable field of vision than is obtained by the ordinary sphero-cylindric equivalent. These lenses, even in the weaker powers, are decidedly advantageous where cemented bifocal lenses are required, since the segments placed upon them for reading are inclined at an angle more closely approaching a position for perpendicular incidence of the visual axes when looking downward.

However, it frequently occurs that a toric meniscus or contrameniscus does not afford the same acuity of vision as its sphero-cylindric equivalent, owing to the fact that their respective principal points do not occupy the same positions before the eye (see description of Figs. 11 and 12). This detriment may, at times, be overcome by slightly modifying the concave spheric surface, the amount being determined by placing a weak spheric lens, between 0.12 and 0.5 (+ or —), in front of the sphero-toric lens while it is being actually used in the test for an improvement of vision.

Heretofore this procedure was unavoidable, and could not be met in any other way, because the spheric and cylindric surfaces of the lenses superposed in the trial-frame and their positions before the eye are not the same as those of the substituted sphero-toric lens. See section on *Vertex-Refraction.*

The focal interval of the toric surface may also be used to advantage in making sphero-toric lenses of the *biconvex form,* more especially in cases where the power of the spheric surface is very considerable, such as required to counteract an aphakia that is associated with astigmatism. An instance in which this type of lens is applied will best serve to illustrate its various advantages.

In examining the refraction in aphakia, it is customary, as in other cases, to place the spheric lens in the groove at the back of the trial-frame, and therefore, nearer to the eye, with the cylinder in front. For example, in a case involving

$$+ 9 \text{ D. sph.} \supset + 3.5 \text{ D. cyl., axis } 160°,$$

the correction is obtained by placing a biconvex spheric lens of + 9 D. in the trial-frame behind the cylinder, + 3.5 D.

The optician would ordinarily execute this prescription *literally,*

by making the lens + 9 D. spheric on one side, and + 3.5 D. cylindric on the other, thereby substituting a *plano-convex* spheric element for the biconvex one used in the trial-frame. Furthermore, in mounting the lens, he would place the spheric surface farther from the eye, with the cylinder inside, thus causing the spectacles to be worn with the surfaces of the lens *reversed* with respect to their positions before the eye in the trial-frame.

It is, therefore, obvious that the serious error is committed of substituting a lens which involves a change in the relative positions of its principal points, thus changing the position of the focus within the eye and to a degree impairing the visual acuity obtained by the spectacles, as compared with that secured by the test in the trial-frame. Aside from this we have, in the aforesaid sphero-cylindric lens, the detrimental phenomenon of *internal reflection,* caused by the extreme bulging forward of the convex element, whose surface is ex-

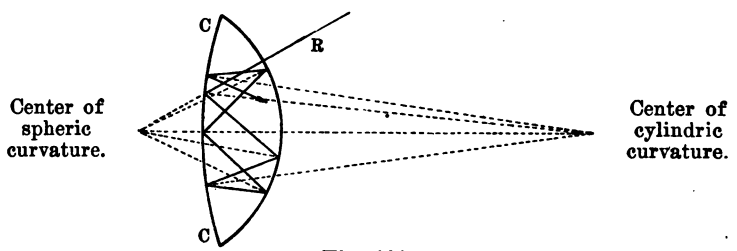


Fig. 101.

posed to light coming from other directions than that of the desired centrally incident beam.

Every ray of light, R, Fig. 101, entering the convex spheric surface in a direction coincident with its radius of curvature, will enter the lens *unrefracted,* and fall upon the inner surface, C, of the cylinder on the other side.

Under favorable conditions such a ray will be reflected from C back to the inner anterior surface of the lens, there again undergoing reflection, and so on, until it becomes dissipated within the lens as diffused light. The dotted lines in the diagram represent the radii of curvature as dividing the angle of incidence and reflection equally at the points of impact on the surfaces within the lens.

Therefore, the person wearing such spectacles is compelled to look, as it were, through a self-luminous medium that tends to interfere with the direct central incident beam of light passing through it to his pupil. Again, in this form of lens the aberration is greater for all excursions of the eye where the visual axis is directed through the lens at points not coinciding exactly with the optic axis of the lens.

These facts were clearly exemplified in a case which came to the author's notice in 1889. The lens had been made of the sphero-cylindric form and power as above stated. The patient complained of inability to correctly judge distances in looking downward, and of an annoying glare of light which he felt was within the lens itself when worn out of doors, or in a strong light. He thought, if the glare could be removed he would be enabled to see better. In fact, he had made the experiment of shading the lens by sighting through the partially closed hand held closely before it, and had noticed less disturbance from the glare.

The sphero-cylindric correction, then worn by him, gave him scarcely better than 6/9 of vision, whereas the test lenses in the trial-frame gave him the somewhat unusual acuteness of 6/6.

It occurred to the author that the interior reflections could be avoided by constructing the lens with less curvature on the anterior surface. Therefore, a lens having a toric surface on the anterior side, and a spheric surface on the other was suggested as follows: + 6 D. sph. combined with a toric surface of + 6.5 D., axis 160° by + 3 D., axis 70°, in place of 9 D. sph. ⌒ + 3.5 D. cyl., axis 160°.

Before proceeding to consider the equivalence of these lenses, it is suggested that all thought of a simple cylindric form should be dispelled from the mind when the toric surface is referred to. The subject will be more easily understood by dealing only with the principal refracting meridians, without regard to the other meridians of curvature that give form to the toric surface.

Referring to the lens, + 9 D. sph. ⌒ + 3.5 D. cyl., it is evident that the meridian of least refraction is 9 D., and the meridian of greatest refraction is 12.5 D., as indicated by m and M in the rectangular planes of Fig. 102.

The extreme spheric curvature, 9 D., on the anterior surface may be lessened to any desired degree, provided the loss of refraction is compensated for by adding it to the opposite cylindric side of the lens, in which case the simple cylindric surface would necessarily be *replaced* by a surface of astigmatic refraction, a toric surface, having the required focal interval of 3.5 D. To obtain such an equivalent lens of *thinnest biconvex form,* it is only necessary to divide the original meridian of greatest refraction, M = 12.5 D., in two parts, as nearly equal as possible; say 6 D., for the back surface, and 6.5 D. for the front surface, which, together, give 12.5 D. for the newly-created *biconvex* meridian of greatest refraction, M_1, Fig. 102, in the desired equivalent lens.

Taking 6 D. as the posterior spheric surface, it is necessary to com-

bine it with 3 D. on the anterior surface to secure 9 D. as the meridian of least refraction, m_1, in the new lens. Consequently, 6.5 D. and 3 D., respectively, represent the refraction of each of the principal meridians of the anterior toric surface, which, when combined with the 6 D. posterior spheric surface, fulfills all the requirements of equivalence, as shown in the sections delineated in Fig. 102.

The sphero-toric lens referred to was set in a spectacle-frame with the toric surface outward, and its meridian of least refraction at 160°, so that the refracting elements occupied the same positions as the lenses in the trial-frame.

The spectacles were worn by the patient for nine years prior to his death, with complete relief from all the disagreeable phenomena men-

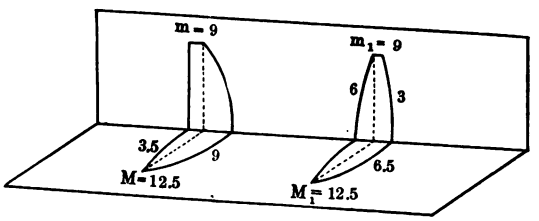


Fig. 102.

tioned. It may also be stated that his visual acuteness with the new lens was slightly better than 6/6.

For reading, the patient required + 12 D. sph. ⊂ + 3.5 D. cyl., axis 160° in the trial-frame, which was given him in the form of a sphero-toric lens: + 8 D. sph. ⊂ toric surface of + 7.5 D., axis 160° by + 4 D., axis 70°.

In addition to the advantages mentioned, the patient was saved the annoyance of wearing uncomfortably heavy lenses that would also attract attention. In every similar instance where the author has applied the sphero-toric lens it has given satisfaction, though in only two other cases has the visual acuteness been so perfect as in the one just cited.

The use of the sphero-toric lens is by no means confined to cases of aphakia, since equally good results can be secured by its use in high degrees of compound myopic astigmatism, especially where the cylindric corrections are weak in comparison with the high spheric curvatures involved. Still greater advantages are to be secured through making one surface of the lens *aspheric*. See section on *Aspheric Lenses of Wide Aperture.*

II. *Contra-generic Meridians.*

While it is also possible to produce a toric surface with *contra-generic* meridians, such as indicated in Fig. 103 and Fig. 104, yet it is not used either separately or in combination with a spheric surface, because its focal interval may be more conveniently utilized through

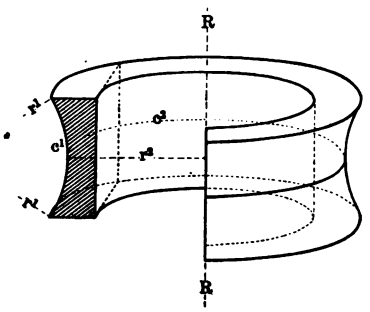


Fig. 103.

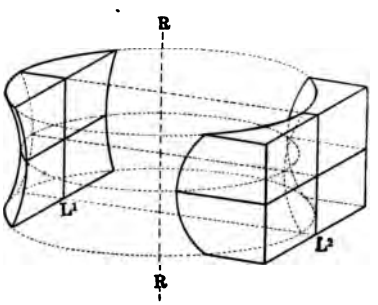


Fig. 104.

combining the usual convex or concave toric surface, shown in Fig. 99, with a suitable spheric surface. The diametrically opposed principal meridians in each of the lenses in Fig. 104 are convex and concave, so that their correlated refractions may, for convenience of illustration, be expressed by:

$$(C_1) \quad [-3 \text{ D. Ref. } 90^\circ \supset\subset +2 \text{ D. Ref. } 180^\circ] \quad \text{Tor. } \ldots . (L_1)$$
$$(D_1) \quad [+3 \text{ D. Ref. } 90^\circ \supset\subset -2 \text{ D. Ref. } 180^\circ] \quad \text{Tor. } \ldots . (L_2)$$

so as to distinguish them from the corresponding formulæ for crossed cylinders of identical refraction:

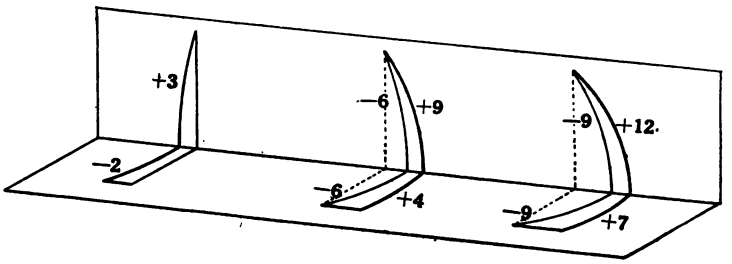
$$(C_2) \quad -3 \text{ D. cyl., axis } 180^\circ \supset\subset +2 \text{ D. cyl., axis } 90^\circ$$
$$(D_2) \quad +3 \text{ D. cyl., axis } 180^\circ \supset\subset -2 \text{ D. cyl., axis } 90^\circ$$

and their sphero-cylindric equivalents, which are, respectively:

$$(C_3) \quad \begin{cases} +2 \text{ D. sph.} \supset\subset -5 \text{ D. cyl., axis } 180^\circ \\ -3 \text{ D. sph.} \supset\subset +5 \text{ D. cyl., axis } 90^\circ \end{cases}$$
$$(D_3) \quad \begin{cases} -2 \text{ D. sph.} \supset\subset +5 \text{ D. cyl., axis } 180^\circ \\ +3 \text{ D. sph.} \supset\subset -5 \text{ D. cyl., axis } 90^\circ. \end{cases}$$

The last mentioned lens is pictured to be equivalent in refraction to the principal sections in Fig. 105, and in which the focal interval of 5 D. is also the numeric difference between the principal meridians of the toric surfaces.

It is sometimes preferable to select a *toric surface* for the deeper and *inner concave surface* of a lens having contra-generic meridians,



Plano-toric. Toric Meniscus. Base-curve, 4. Toric Meniscus. Base-curve, 7.

Fig. 105.
Contra-generic Meridians.

the choice being usually governed by the nature of the visual defect that it is intended to counteract. In other words, the curvatures of the toric lens should as closely as possible harmonize in kind and position before the eye with those occupied by the sphero-cylindric lens in the trial-frame.

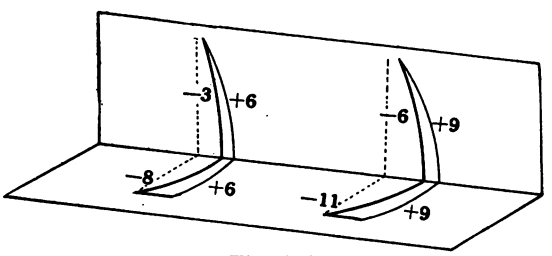


Fig. 106.
Concave Toric Surfaces.

For instance, the sections in Fig. 106 make it clear that the 6 D. and 9 D. spheric curvatures may be made convex instead of concave, through combining them with *concave* toric surfaces that should be placed next to the eye, the same as would be the case if a lens + 3 D. sph. ⌒ − 5 D. cyl. were used, and in which the concave cylindric element is worn near the eye.

Wide-Aperture Lenses.

In order to satisfy the stipulation that the focus of an ophthalmic lens shall coincide exactly with the far point of the ametropic eye, it is evident that two lenses of materially different forms (bispheric

and meniscus or contrameniscus), placed alternately at the same distance from the eye, must differ in power, because the distance between their respective focal and principal points is then not the same dimension. For instance, the thin meniscus, whose principal points are slightly beyond the outer surface of the lens, and therefore farther from the far point of the hyperopic eye, should be correspondingly weaker than the biconvex trial-lens; whereas, the thin contrameniscus, having its principal points somewhat nearer than the inner surface of the lens and closer to the far point of the myopic eye, must be a little stronger than the biconcave trial-lens. See section on *Vertex-Refraction.*

For this reason the representatives of Carl Zeiss in Jena, Germany, the Bausch & Lomb Optical Co., have in America introduced a more exact system, devised by Prof. Moritz von Rohr, through which the dioptral powers of ophthalmic lenses are expressed in vertex-refraction (D_v), as reciprocals of their focal lengths, which are counted from the vertices of their back surfaces, irrespective of typical lens-form. This principle applies advantageously to all lenses of wide aperture and especially to the so-called Punktal lens, which is numbered in vertex-refraction and designed to eliminate the astigmatism due to oblique light-incidence and to render available a field of view of about 60°, in order that objects viewed through the extreme marginal zone of the lens may be seen as distinctly as through its center. In other words, each point of the object is faithfully reproduced as a distinct point in the image formed by the Punktal lens, which is either a meniscus or contrameniscus, whose concave surface is different for each power between 0.25 D_v and 20 D_v, the other surface being either spheric or toric. However, there is a notable exception, namely, when the power of the *convex* Punktal lens exceeds 7 D_v it is called a Katral lens, in which the concave surface is made *aspheric,* in order to correct the aberration for the full aperture of a meniscus of high power. Punktal inner curves vary by 0.0012 meters and sometimes less, being ground and polished with the same care as a photographic objective, so that, if bifocal lenses are desired, they can only be made in the fused type. Similarly devised lenses are being exploited in Germany, by different makers, under the names, Rectaviz and Isokryster, respectively.

Punctilious application of vertex-refraction entails provision being made for determining the exact distance between the cornea of the eye and the back surface of the trial-lens in a trial-frame of special construction; this being supplemented by an instrument, vertex dioptrometer, which accurately measures the lens of any type that may be

subsequently used to substitute the lens or lenses in the trial-frame. To facilitate this substitution without the measuring instrument tables are supplied through which may be ascertained the vertex-refraction of the ophthalmic lens that is required to be worn at a prescribed distance from either the myopic or hyperopic eye, because such indicated powers are dependent upon the distance between the cornea and the proximate vertex of the accurately adjusted trial-lens. As this interpolar distance naturally varies with the facial anatomy peculiar to different individuals, it is necessary in the tables to give it specifically different values from 4 mm. to 20 mm. If the correcting lens is to be a meniscus or contrameniscus, allowance must also be made for the vertex-depth of its concave surface, since the pole of this lens, with respect to the interpupillary plane of the frame, will be farther from the eye than the pole of the lens used in the trial-frame, and from whose vertex the interpolar distance is measured.

Therefore, still another table of corrections is supplied, in which the vertex-depths, in millimeters, are given for a series of concave surfaces, varying in power between — 1 D. and — 20 D., upon lenses whose diameters vary by millimeters between 34 mm. and 46 mm. In fact, strict compliance with this table is of greatest moment when prescribing menisci and contrameniscі or Punktal and Katral lenses whose powers exceed 3.5 D_v. None of these lenses admit of being neutralized by each other, or by lenses of other types.

Reverting to the first mentioned tables we find 17 different dioptral powers (D_v) of vertex-refraction are introduced as equivalents for each of the powers of the ordinary dioptral lenses that are placed at the same number of distances from the cornea.

TABLE I

Myopic Eyes

Axial Refraction in Diopt.	Vertex Refraction of Correcting Lens for a Distance in Millimeters between Vertex of Lens and Vertex of Cornea Equal to															
	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
— 3.00	3.05	3.05	3.05	3.10	3.10	3.10	3.10	3.10	3.10	3.15	3.15	3.15	3.20	3.20	3.20	3.20	3.20
— 3.25	3.30	3.30	3.30	3.35	3.35	3.35	3.40	3.40	3.40	3.40	3.40	3.40	3.40	3.45	3.50	3.55	3.60
— 3.50	3.55	3.60	3.60	3.60	3.60	3.65	3.65	3.65	3.70	3.70	3.70	3.70	3.70	3.75	3.75	3.80	3.80
— 3.75	3.85	3.85	3.85	3.85	3.90	3.90	3.90	3.95	3.95	3.95	3.95	4.00	4.00	4.00	4.05	4.05	4.05
— 4.00	4.10	4.10	4.10	4.10	4.15	4.15	4.20	4.20	4.20	4.25	4.25	4.25	4.30	4.30	4.35	4.35	4.35
— 4.25	4.35	4.35	4.40	4.40	4.45	4.45	4.45	4.50	4.50	4.50	4.55	4.55	4.60	4.60	4.60	4.65	4.70
— 4.50	4.60	4.65	4.65	4.70	4.70	4.75	4.75	4.75	4.80	4.80	4.85	4.85	4.90	4.90	4.90	4.95	5.00
— 4.75	4.90	4.90	4.95	4.95	5.00	5.00	5.00	5.00	5.10	5.10	5.10	5.15	5.20	5.20	5.25	5.30	5.30

TABLE II

Hyperopic Eyes

Axial Refraction in Diopts.	Vertex Refraction of Correcting Lens for a Distance in Millimeters between Vertex of Lens and Vertex of Cornea Equal to																
	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
+ 3.00	2.95	2.95	2.95	2.95	2.90	2.90	2.90	2.90	2.90	2.90	2.90	2.85	2.85	2.85	2.85	2.85	2.80
+ 3.25	3.20	3.20	3.20	3.15	3.15	3.15	3.15	3.10	3.10	3.10	3.10	3.10	3.10	3.10	3.05	3.05	3.05
+ 3.50	3.40	3.40	3.40	3.40	3.35	3.35	3.35	3.30	3.30	3.30	3.30	3.30	3.30	3.30	3.25	3.25	3.25
+ 3.75	3.70	3.65	3.65	3.65	3.60	3.60	3.60	3.60	3.55	3.55	3.55	3.55	3.50	3.50	3.50	3.50	3.50
+ 4.00	3.90	3.90	3.90	3.90	3.85	3.85	3.80	3.80	3.75	3.75	3.75	3.75	3.75	3.70	3.70	3.70	3.70
+ 4.25	4.15	4.15	4.10	4.10	4.10	4.10	4.05	4.05	4.05	4.00	4.00	4.00	4.00	4.00	3.95	3.90	3.90
+ 4.50	4.40	4.40	4.35	4.35	4.30	4.30	4.30	4.25	4.25	4.25	4.20	4.20	4.20	4.15	4.15	4.15	4.10
+ 4.75	4.65	4.65	4.65	4.60	4.55	4.55	4.50	4.50	4.50	4.50	4.50	4.40	4.40	4.40	4.35	4.35	4.35

TABLE III

Vertex-Depths of Ophthalmic Lenses

Curvature of the Hollow Surface of Lens	For Diameters in Millimeters Equal to												
	34	35	36	37	38	39	40	41	42	43	44	45	46
− 3.00	0.9	0.9	1.0	1.0	1.1	1.1	1.2	1.2	1.3	1.4	1.4	1.5	1.5
− 3.25	1.0	1.1	1.1	1.2	1.2	1.3	1.3	1.4	1.4	1.5	1.6	1.6	1.7
− 3.50	1.1	1.2	1.3	1.3	1.4	1.4	1.5	1.5	1.6	1.7	1.7	1.8	1.8
− 3.75	1.1	1.4	1.4	1.3	1.4	1.5	1.6	1.6	1.7	1.8	1.9	1.9	2.0
− 4.00	1.2	1.2	1.4	1.4	1.5	1.6	1.6	1.7	1.7	1.8	1.9	2.0	2.1
− 4.25	1.2	1.3	1.3	1.4	1.5	1.6	1.7	1.7	1.8	1.9	2.0	2.1	2.2
− 4.50	1.3	1.4	1.5	1.5	1.6	1.7	1.8	1.8	1.9	2.0	2.1	2.2	2.3
− 4.75	1.4	1.5	1.6	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5

These tables are abbreviations of the originals covering the full range of powers between 0.25 and 20 dioptries, as published by the Bausch & Lomb Optical Co.

For instance, the regular trial-lens of + 4 D., whose pole is 4 mm. from the cornea, is equivalent to + 3.9 D_v, whereas, if placed at 12 mm., it is + 3.75 D_v, or, if at 20 mm., it is + 3.7 D_v. Consequently, in order to secure the full benefit of this exploited precision, accurate measurement of the actual interpolar distance is of the utmost importance, if indeed the lens that precisely corresponds to it can be supplied within the range of dioptral values given in the tables as equivalents and needful substitutions in vertex-refraction, and which would, of course, represent an unprecedented degree of precision among ophthalmic lenses.

However, for the present the range of powers in vertex-refraction for Punktal lenses is being confined to quarter-dioptry intervals, so that the actual vertex-refraction of the ordinary trial-lens of a given number will in practice have to be substituted by the nearest avail-

able power of a Punktal lens that is to occupy the same position given in the tables, I and II. Still, this *method* of substitution may be conveniently avoided through the use of the vertex-dioptrometer, which measures the dioptral refraction from the vertex of the lens instead of from the principal point.

Both surfaces of the Punktal lens are mathematically calculated upon the assumptions that the back surface of the lens is 25 mm. from the center of rotation of the eye-ball, and that the lens and the refractive media of the eye together constitute an optical system in which the cross-section of the refracted pencil of light at this center of rotation is the exit-pupil of the system. This motive being strictly complied with makes it evident that the relative proportions to each other of the front and back surface-curvatures necessary to provide a range of powers in Punktal lenses can only in very exceptional instances, if any, be the same as those of the ordinary menisci and contramenisci having standard base-curves of 6 D. or 9 D. and which, "because of their similar correction of astigmatism in the marginal parts, have a useful field of view of 52°." Therefore, it remains to be seen whether a greater variety of well chosen base-curves applied to the latter may not after all in some instances, by way of substitution, suffice to produce an inappreciable difference when such improved wide-angled lenses are practically applied, especially as the fields of view of similar lenses are now only 4° less at the margin than in Punktal lenses. In fact, this proposition would merely involve "bending the lens" in strict accordance with the formulæ known to physicists (*Principles of Geometrical Optics.*—Southall) long before the successful attempt was made by Dr. M. von Rohr to correct the aberration in lenses of high power by means of the *aspheric* surface, and which is now exclusively applied to and admitted to be necessary only in menisci and contramenisci above 7 D_V. See section on *Aspheric Lenses of Wide Aperture.*

In order to meet this emergency the Bausch & Lomb Optical Co. has engraved its trade-mark upon each Punktal lens, and, indeed, so faintly that it requires a special instrument to detect it. Although in a few typical forms and powers the Punktal lens may be objectionably thick, it is unquestionably the most scientifically devised ophthalmic lens as yet produced.

Aspheric Lenses of Wide Aperture.

Dr. Moritz von Rohr, physicist, of the firm of Carl Zeiss, Jena, Germany, at the suggestion of the ophthalmologist, Dr. A. Gullstrand, Upsala, Sweden, mathematically determined the curvatures of the so-

called Gullstrand cataract lens, which is provided with a concave surface that is more acutely curved near its periphery, a spheroid surface, which Dr. von Rohr calls an *aspheric surface,* whose function it is to counteract the astigmatism produced by the oblique ray-bundles that would otherwise enter the eye when the visual axis is directed through the peripheral parts of a spheric lens of high power, and such as is required to correct the refraction in aphakia, whether paraxial astigmatism is present, or not.

Lenses of this type, from $+8\ D_v$ upward, being designed for surgically operated cataract cases, are now being sold by the Bausch & Lomb Optical Co. as Katral lenses, which have the same optical properties as Punktal lenses. Either of these lenses provides an unrestricted field of 60°, permitting the wearer to rotate the eyes in a normal manner and to secure distinct vision even through the margins of the lens.

In a Katral lens of $12\ D_v$ the radius of the osculating sphere is actually 140 mm., and the difference of the aspheric surface is only 0.16 mm., which shows the extreme precision applied in their manufacture.

At a meeting of the Ophthalmological Society in Heidelberg, 1910, von Rohr exhibited photographic plates which demonstrate that Gullstrand's cataract lens, in this instance an *asphero-toric meniscus,* permits excursions of the visual axis from the pole of the lens toward its periphery to the extent of 30° in one of its principal sections, and 36° in the other, with greater freedom from aberration than is obtained for a similar excursion of 12° behind a sphero-cylindric lens intended to correct 13 D. and 7 D. of refraction in the principal sections of the eye that are inclined to each other at an angle at 45°.

However, applications of the Katral lens involve intricate calculation being made for its construction, the same being based upon definite information respecting the distance between the cornea of the eye and the proximate surface-vertex of the inner trial-lens in the trial-frame, either in case a spheric or sphero-cylindric lens is used in it for distant vision, or for reading at a specified distance. Similar specifications are required in order to calculate the curvatures of a contrameniscus for extreme myopia. The cost of each Katral lens is listed between $20.00 and $55.00, the price depending upon its typical form and power.

Incidentally, it may be mentioned that aspheric surfaces have been produced at the Zeiss works ever since 1899, yet their applications were later confined chiefly to the finer wide-angled condensers for the substage of the compound microscope, and in 1910 von Rohr incor-

porated the aspheric surface in the aplanatic lens used in Gullstrand's ophthalmoscope, which is constructed especially to obviate the extraneous light-reflections from the observed eye. In 1908 von Rohr gave credit to his colleague, O. Henker, for having perfected the mechanical agents through which Gullstrand lenses were produced and photographically tested to prove their efficiency.*

Aberration.

The desired deviation which rays of light sustain in their passage through a lens is exposed to two principal faults that prevent the refracted rays from meeting in a common focus, thus producing a distorted or indistinct image that is also associated with more or less color. The latter phenomenon is called *chromatic aberration,* and is due to the decomposition of white light produced by the inherent variable prismatic power of the lens, which is greater near its edge.

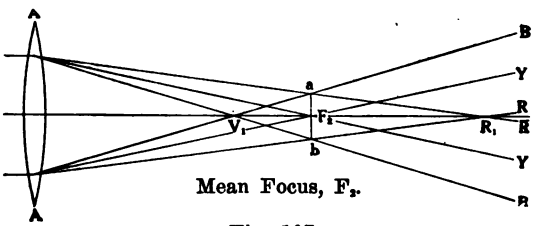


Fig. 107.
Circle of Chromatic Aberration, ab.

The violet rays of the spectrum, being more refrangible than the red rays, cause the former to cross the optic axis nearer to the lens, so that the more remote focussed image is fringed with color, usually blue.

The focal length of the lens is shorter for blue rays than for red rays, or the two principal foci, F_1 and F_2, for blue rays, B, are nearer to the lens than those for red rays, R, Fig. 107. Nevertheless, a lens may be constructed so as to be tolerably free from this defect, when it is called an *achromatic lens* (see *The Fundamentals of Achromatism*). It commonly consists of two superposed spheric convex and concave lenses, usually united through balsam, and respectively made of glass

* Die Theorie anastigmatischer Starbrillen, M. von Rohr.—Bericht ueber die XXXV Versammlung der Ophthalmologischen Gesellschaft, Heidelberg, 1908.
Ueber Gullstrandsche Starbrillen mit besonderer Beruecksichtigung der Korrektion von postoperativem Astigmatismus, M. von Rohr.—Bericht ueber die XXXVI Versammlung der Ophthalmologischen Gesellschaft, Heidelberg, 1910.
Ueber neuere Bestrebungen in der Konstruktion ophthalmologischer Instrumente, M. von Rohr.—Bericht der XXXVII Versammlung der Ophthalmologischen Gesellschaft, Heidelberg, 1911.

having different refractive and dispersive powers, such as crown-glass and flint-glass, with their curvatures so accurately proportioned that the chromatic aberration produced by one lens is counteracted by that of the other lens. However, in ophthalmic lenses, chromatic aberration is not so manifest as *spheric aberration,* which consists of both *longitudinal* and *lateral aberration,* due to a curvature of the lens which causes those rays that are refracted near its edge to cross each other before reaching the more or less sharply defined focal point, F_2, Fig. 108, of the central rays. These consecutive points of intersection, p_1, p_2, etc., generate in any principal axial plane a caustic curve whose cusp is at the focus, F_2. Consequently, the entire bundle of refracted rays is enveloped by a surface called the *caustic,* which is the evolute of the refracted wave. Even actual point-images which are necessarily

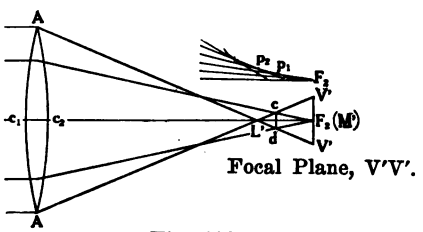


Fig. 108.
Least Circle of Spheric Aberration, cd.

formed by bundles of rays of finite aperture (a narrow zone immediately surrounding the optic axis) are, in general, more or less associated with longitudinal aberration that is not discovered by the eye, owing to its comparatively poor resolving power. Furthermore, the rays of an isolated bundle, obliquely incident upon a very small surface-element at the peripheral part of the lens, are transformed by refraction, at a surface of any form, into a non-homocentric or astigmatic bundle of rays, all the rays of which, at least to a first approximation, intersect two definitely short lines, the so-called image-lines of the bundle. This phenomenon was referred to under *Astigmatism,* and which, under suitable conditions, was also shown to be inherent in any simple lens. The longitudinal aberration is measured along the optic axis, being the distance between the focus, F_2, of the paraxial rays and the nearer point of axis-intersection, L′, of the rays that emerge from the periphery of the aperture, and upon whose magnitude, $L' F_2$, the aberration depends. The longitudinal aberration in thin lenses of the same focal length varies with their form; in other words, it depends upon the curvatures, c_1 and c_2, given to the lens. Since it is not practically feasible to entirely abolish longitudinal

aberration in an infinitely thin lens that is exposed to an object-point situated at infinity, and in order to secure at least the minimum aberration in a thin lens so exposed, the curvatures c_1 and c_2 must bear the following relation (*Principles of Geometric Optics,* Southall) to each other:

$$c_1 : c_2 = \frac{n(2n+1)}{2n^2 - n - 4} \quad \text{.....XV.}$$

$$\text{when } c_1 = \frac{n(2n+1)}{2(n-1)(n+2)} f, \text{ and } c_2 = \frac{2n^2 - n - 4}{2(n-1)(n+2)} f, \text{ XVI.}$$

wherein n is the refractive index, f is the focal length, and $c_1 = \frac{1}{r_1}$ and $c_2 = \frac{1}{r_2}$ are the curvatures of the bounding lens-surfaces whose radii are r_1 and r_2. Within certain rational limits it is also feasible to vary the curvatures of the opposite surfaces of a lens without altering its focal length; this procedure being called "bending the lens." For instance, it is "bending the lens" to give a lens of given focal length such typical form as shall correct spheric aberration throughout its entire field, as in the Punktal lens. Through the above equations, and the choice of an index = 3/2 it can be shown that a biconvex or biconcave lens has the least longitudinal aberration when the curvature of its front surface, c_1, is six times as great as the curvature, c_2, of its back surface (the greater curvature having the shorter radius). This lens is called a "crossed lens."

The plano-convex lens is nearly as good as the crossed lens whose aberration is only 2 or 3 per cent. less than the former, is easier to make, and therefore, more commonly used in eye-pieces, etc. The crossed lens of flint-glass, with a refractive index of 1.6, is a plano-convex lens. The *least circle of aberration* is perpendicular to the axis and located within the amplitude of the longitudinal aberration. Its diameter is determined by the intersections of the edge-rays with the caustic surface after these rays have crossed the axis. The distance of the least circle of aberration, cd, from the image-point is approximately equal to ¾ of the longitudinal aberration of the extreme outside ray.

The radius, M′ V′, of that circle inside of which all the rays of the bundle cross the perpendicular image-plane at the cusp, and which

is produced by intersections of the extreme edge-rays with the image-plane, is called the *lateral aberration.*

Spheric aberration may also be more or less obviated by means of an aperture-stop placed in close proximity to the lens so as to exclude the peripheral rays that would otherwise interfere with the central rays in their production of a distinct image. In fact, this is accomplished in the construction of every compound lens-system through introducing an annular disk of *calculated* diameter, known as the *diaphragm* or stop, which is suitably placed between the lenses to effectually exclude the peripheral rays. If the proper diaphragm be replaced by one of smaller aperture, we increase the definition, but diminish the extent of field and illumination. A larger aperture will increase illumination and field, but definition will be impaired, on account of the aberration thus allowed.

The aperture of the diaphragm must, therefore, have a definite and specific diameter in every optical instrument, if we are to secure *maximum* definition and illumination, *without aberration.* The proper diaphragm is, therefore, one of the most important and indispensable parts of every compound dioptric system.

The following description of ocular aberration is given, because the aberration of the eye itself should be explained in an essay which aims to describe the lenses used to remedy its various optical defects.

Ocular Aberration.

The human eye, being a compound dioptric system, is provided with its diaphragm—the iris. In the eye, which is a dynamic apparatus given to variations of power, a fixed diameter of pupil would fail to theoretically fulfill the requirements. When the eye is in a state of accommodation, it becomes a stronger refracting system, and therefore needs a smaller aperture of diaphragm, hence the pupil contracts.[1] Yet, Helmholtz [2] says: "A. von Graefe observed in an eye from which he had removed the iris by operation that the normal range of accommodation was still present, and also that the changes in the anterior curvature of the lens could still be observed." He concludes: "The iris, therefore, does not play an important rôle in accommodation" (lit. trans.). Landolt [3] expresses the same opinion. While the measurements referred to are undoubtedly correct, yet, if the conclusion

1. In fact, it was at one time supposed that contraction of the pupil was the only means by which the eye adapted itself for near vision. Helmholtz, *Physiologische Optik,* page 151, Hamburg and Leipzig, 1886.
2. Helmholtz, *Physiologische Optik,* page 138.
3. Landolt, *Refraction and Accommodation of the Eye,* page 164, Philadelphia, 1886.

reached is construed in its broadest sense, it discountenances the value of the iris as a diaphragm entirely.

It is, nevertheless, universally admitted that the iris does act independently of, yet simultaneously with, accommodation.[4] When acting independently of accommodation, the iris is known to behave as a highly sensitive photostat,[5] through regulating the volume of light upon the retina to such a degree as shall be most agreeable to the light-perceptive sense.

A most subtle and synchronous balance, between retinal perception, uveal stimulus and iritic response, must, therefore, exist, if the iris is to perform its functions simultaneously as diaphragm and photostat.

An endeavor is here made to support the hypothesis that *a disturbed equilibrium of these functions is probably the cause of asthenopia in low degrees of ametropia.* From a strictly optical point of view every eye of the same refraction, other things being equal, should have a pupil of the same diameter—one suited, by *calculation,* to exclude peripheral aberration, while securing the greatest tolerable illumination. This, however, is not known to be the case, nor has the author found that any one has ever calculated what the diameter of the pupil should be for any given schematic eye. Listing has calculated a table showing the changes in diameter of the diffusion circles upon the retina which arise through efforts of accommodation in a schematic eye having a pupil of 4 mm. (Helmholtz, *Physiologische Optik,* page 127.)

We have thus far been content to know that pupils differ in size in different persons. There must, however, be a *limit* to the maximum diameter of the pupil, if aberration is to be excluded, and if, *for any reason,* the pupil is prevented from contracting to at least this limit, we shall have aberration and an associated impairment of vision, even in the emmetropic eye.

This is exaggeratedly shown in Fig. 109, in which the central incident rays, cc, focus at f upon the retina, while the peripheral rays, pp, produce thereon an area of diffusion,† ab, and which, to all practical purposes, would be equally as effective in impairing vision as a low

4. "Movements of the iris are nevertheless associated with accommodation; they are governed by the same nerves as the latter, so that, until the mechanism of accommodation is better understood, a direct relation between them may not be looked upon as being improbable" (lit. trans.).—Donders, *Refraction and Accommodation,* page 485, Wien, 1866.

5. Photostat, Greek Φῶς (Φωτ-) light, + ατατός, verbal adjective of ἱστάναι, stand—an automatic light regulator, suggested by the author in "The Iris, as Diaphragm and Photostat."—*Annals of Ophthalmology and Otology,* October, 1895.

† For purposes of lucid illustration, the diffusion areas in all of the diagrams are greatly exaggerated.

degree of myopia, having its intra-ocular focus anywhere between the retina and f_1. Therefore, *it is evident that the instillation of a mydriatic, even in the emmetropic eye, also produces a defect which is not to be mistaken for a lenticularly corrigible ametropia.* In fact, it is questionable whether the eye can discriminate between images which are impaired by peripheral aberration and those that are illy defined through slight errors of refraction. The following experiment will serve to illustrate this. By placing a 1 D. convex lens before the emmetropic eye, it is practically rendered myopic for distance, the letters of the test-card at 6 m. becoming indistinct, with a probable reduction in the visual acuteness to, say, 6/9. If the lens be now covered with a pin-hole disk, normal acuteness of vision will be re-

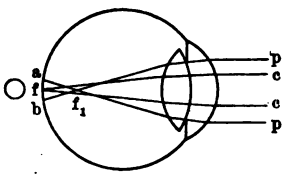


Fig. 109.

established, with no other appreciable difference than that the field and illumination are less. We may, therefore, consider the peripheral rays, here accompanying the increased refraction, as aberrative rays with respect to the enclosed central incident beam, so that an eye capable of contracting its pupil to the same extent would, in part, similarly correct its error of refraction.

This is undoubtedly one reason why errors of refraction of the same degree are not accompanied by the same diminution of visual acuteness in different individuals. The myope of 1 D., with small pupils, *without* glasses, will probably have better vision than the myope of 1 D. with much larger pupils. Within certain limits, peripheral aberration and anomalies of refraction are analogous in destroying definition of the image. A slight error of refraction, with large pupils, may produce diffusion circles equally as pronounced as a considerable refractive error with small pupils.

Asthenopia is, therefore, quite as apt to be experienced on account of the size of the pupil as it is on account of the error of refraction.

This should explain why it is that many persons, having small pupils, endure a considerable error of refraction without inconvenience, whereas others, with large pupils and small errors of refraction, are afflicted with asthenopia. Again referring to Fig. 109, the larger the pupil the greater will be the zone of peripheral aberration and its correlated diffusion-area, ab. In fact "the peripheral aberration

upon the optic axis is known to increase, not only in proportion to the square of the aperture, but, also *pari passu* with the refraction" (physical law), so that we should have greater diffusion circles upon the retina, when the ciliary muscle is brought into action, even in emmetropia, to correct the peripheral aberration that impairs the sharp definition at f. The only stimulus that could assist in correcting the aberration in this case would be that which, imparted to the

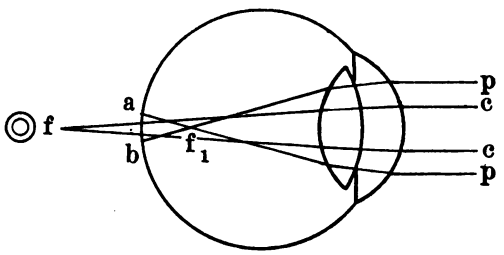


Fig. 110.

iris from the retina, would cause the pupil to contract sufficiently to exclude peripheral rays. In here speaking of the retina, we of course take for granted its highest state of physiologic development. The question then arises: In low degrees of ametropia, is such stimulus imparted to the iris *independently of accommodation, without increased light intensity?* If there is such independent action on the part of the iris, ineffectual efforts of the ciliary muscle to correct

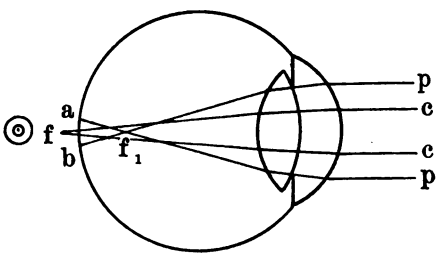


Fig. 111.

impaired vision may be followed by a contraction of the pupil necessary to shut out the peripheral rays. As to this, let us investigate the relation that should exist between the iris and accommodation in the slightly hyperopic eye.

In this case, Fig. 110, the central rays, cc, are focused behind the retina at f, the peripheral rays crossing at f_1 to produce the diffusion-area, ab. In *facultative hyperopia* there will be accommodation sufficient to bring f forward to the retina. With this increased re-

fraction, however, the pupil remaining the same, f_1 will recede from the retina, with a corresponding increase in the size of the diffusion-area, ab.* It is, therefore, evident that, if *increased* aberration is to be avoided, a *normal* pupil must contract concurrently with the accommodation. This, generally speaking, is known to be the case. If, as in Fig. 111, the hyperopia is of low degree, with an *excessively* large pupil, we shall have a comparatively small central area of diffusion, due to the refractive error, covered by a much larger area of diffusion and illumination, ab. The slightest effort of accommodation would tend to sustain or increase this discrepancy. Therefore, it follows that, if the aberration is to be abolished, the iris must receive an increased stimulus to bring about a contraction of the pupil *in excess of that which is concurrently associated with accommodation, and that, too, for every change in light intensity.* Were this not the case, vision

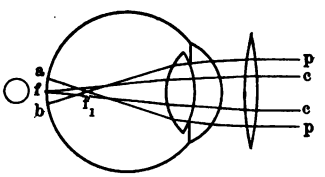


Fig. 112.

for distance,† with excessively large pupils, would be impaired by aberration under all circumstances.

The additional stimulus to contraction is undoubtedly due to the *increased area of illumination* above mentioned. This would seem to imply that the contraction of the pupil not only responds to the light intensity (quality), but also to its area (quantity) upon the retina.

It is also evident that the impairment of vision should be ascribed to that factor causing the greater area of diffusion upon the retina. The larger the pupil, the more will the peripheral aberration predominate over that which is produced in the center by a low degree of refractive error.

By placing the correcting lens before the eye in hyperopia we increase the refraction, thus eliminating the diffused central image while at the same time increasing the peripheral aberration, and,

* Listing's table shows that the diffusion circles upon the retina increase more rapidly as the object approaches the eye at short range.—Helmholtz, *Physiologische Optik*, page 128.

† In accommodation, with a standard light placed behind the plane of the eyes, and an approach to them of the paper upon which the test-type is printed, the illumination upon the paper increases in the inverse proportion to the square of the reduced distance between the light-source and the test-object. The illumination also varies directly as the cosine of the angle of incidence upon the illuminated surface.—Physical law of photometry.

therefore, also the *area of illumination,* Fig. 112. If the pupil contracted only in *proportion* to the consequent increased light-stimulus, there would still remain the original diameter of diffusion area. As, however, correction of the refractive error by the lens improves vision and relieves asthenopia—being *tacit proof* that the aberration is dispelled—it is evident that the pupil must contract *more* than in proportion to the aforesaid light-stimulus. Is it not then probable that the pupil contracts *more freely* when accommodation is relaxed?

In controverting this, it would be necessary to refute the following fact pertaining to combined kinetic energies:

When accommodation is in force, the iris is known to be carried forward (Helmholtz, *Physiologische Optik,* page 131, Wien, 1886) by pressure from the anterior surface of the lens, which has become more strongly curved. Such lens-pressure, *the iris remaining inactive,* would tend to incease the diameter of the pupil. On this account, greater efforts of the sphincter will be necessary to counteract this action of the lens-surface when accommodation is present, Fig. 113, than it would with relaxed accommodation, Fig. 114.

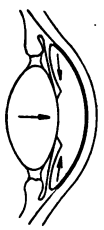


Fig. 113.

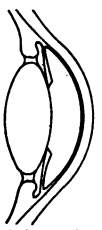


Fig. 114.

However, for normal conditions of innervation the sphincter is known to more than overcome such action on the part of the lens in accommodation, Fig. 113. If, therefore, our hypothesis is correct, we have found a reason why low-degree lenses are of so much benefit in slight hyperopia and congeneric astigmatism. Furthermore, we are justified in assuming that the sphincter in large pupils does not always adequately respond, *while accommodation is in force,* especially in cases where the optical error is so slight as a quarter-dioptry, from the fact that, in the majority of such cases the patients are young and often possesses amplitudes of accommodation varying between 6 and 14 dioptries.

Patients with such accommodation have so much of it in reserve, even when using the eyes in proximity, that their asthenopia can scarcely be ascribed to an overtaxed ciliary muscle. Are we not then justified in attributing it to possible fatigue of the iris, resulting from

its involuntarily prompted, though futile, efforts to exclude peripheral aberration, on account of the sphincter's inability, *for some reason,* to contract sufficiently?

It is not recorded that a disproportion of the pupils to the dioptric system of the eyes does ever exist physiologically, but there are many conditions of the nervous system which produce immoderate dilatation of the pupils. Such dilatation, *while it lasted,* would tend to oppose the normal association between refraction and the correlated size of the pupil.

In those cases of *normal* pupil, where the perceptive function of the retina is keen and the error of refraction is slight, retinal stimulus will prompt contraction of the pupil sufficiently to exclude aberration. Is it not probable that, in some cases with *large* pupils, protracted efforts of this kind would result in fatigue of the iris? Might not prolonged ineffectual efforts of the iris to regain equilibrium between its functions, as diaphragm and photostat, account for asthenopia? Or, to put it in another way: Could not that prolonged effort of the sphincter, which would have to be *in excess* of the normal qualitative and quantitative light-stimulus to correct aberration, produce asthenopia?

It need not follow that the iris is incapable of temporarily contracting even to a greater extent than is necessary for the above purpose. This is demonstrated by the extreme contraction of which the pupil is generally capable when exposed to intense light, and the eye is in its static state of refraction.

In hyperopes, we generally ascribe the cause of asthenopia to fatigue of the ciliary muscle, owing to its efforts to exclude the error of refraction by accommodation. The same can not be said of myopes, whose use of accommodation for such purpose would only render them deplorably more myopic. Their asthenopia can certainly not be ascribed to ciliary fatigue. Some myopes, however, endeavor to improve their vision by compressing the eyelids, which means that they thereby *modify the pupils* to exclude peripheral rays and the aberration which is heightened by the myopia. In low degrees of myopia and congeneric astigmatism, however, modification of the pupils, by compression of the eyelids, is not sufficiently *delicate* to exclude aberration, *without too great a sacrifice of illumination.* Such patients are, therefore, more apt to apply for relief from glasses than those who help themselves by compression of the eyelids, provided this is unaccompanied by asthenopia. In the former cases, we are to suspect that the relief sought is *freedom from peripheral aberration.* This aberra-

tion also aggravates photophobia, which is a symptom frequently complained of in such instances.

The improvement in vision which the myope, of low degree, with large pupils, secures by the lenticular correction, is practically due to the fact that the peripheral aberration is decreased, through the reduced refraction obtained by the concave lens in front of the eye, Fig. 115.

The rays emitted from the concave lens enter the pupil with a divergence counteracting the excessive convergence of the rays which are imperfectly focused by the crystalline on the retina behind f. The peripheral diffusion-area, ab, may not, however, always be in such proportion to the central diffusion-area as to be fully corrected by the lens which corrects the refractive error in the center. Should it, in the case of a larger pupil, be greater, the patient would merely then

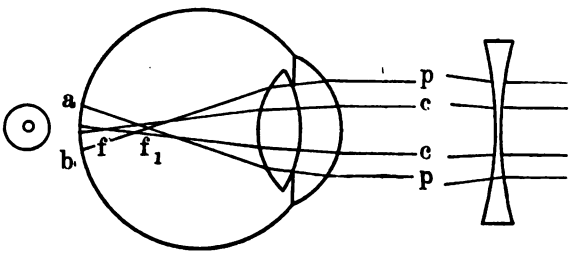


Fig. 115.

select a stronger lens having its proper effect upon peripheral rays, while its tendency would also be to overcorrect the myopia. For a low degree of myopia this would scarcely be appreciable, since very little difference in refraction is experienced in the actual centers between lenses of a quarter and a half-dioptry.

In those cases where the quarter-dioptry lens seems to relieve asthenopia it will generally be found that the pupils are comparatively large. This is especially noteworthy in those cases where myopes with lenses of trivial power are benefited by wearing their distance corrections for *reading,* and which can serve no other *needful* purpose than to eliminate peripheral aberration.

So far, we have no means of ascertaining the size of the pupil, or that variation of it, which is necessary to establish the proper harmony between refraction, accommodation, illumination and freedom from aberration. The intuitive discrimination, which accompanies experience, is at present our only guide.

In refractive errors of low degree, which are relieved by lenticular correction, the retinal perception is usually also very keen, thus

increasing stimulus to contraction of the sphincter, while the correction in such cases frequently improves vision to 6/3, which is far above the conventionally accepted normal acuteness of vision.

The larger the pupil, the more pronounced will be the improvement in visual acuteness obtained by low-degree corrections. The quarter-dioptry lens rarely proves of benefit when the pupils are small.

Again, patients frequently wear such glasses, which for a time relieve the asthenopia, and ultimately lay them aside, without feeling the necessity of their further use. Examination will nevertheless reveal the fact that *the optical error has not changed.* Why then should asthenopia exist at one time, and not another, for an *invariable hypermetropic astigmatism,* for instance, if the fatigue in the first instance had only been due to that of the ciliary muscle?

Closer examination, however, will frequently show that the pupils appear to be smaller at the time the patient has discarded his glasses than when they were prescribed. The pupil being the only member seeming to have undergone a change, are we not justified in suspecting the iris, by reason of disturbed innervation, as having been at least implicated in the cause of asthenopia?

It is commonly understood that visual acuteness depends upon the perceptive functions of the retina, as well as upon the size of the image projected upon it, and which is limited by the visual angle subtended by the object at the ocular nodal point. Therefore, to the exclusion of all other considerations, the ability to discern objects primarily depends, first, upon contrast between light and shade, involving also the color sense; and, second, upon the size of the object viewed. Hence, the greater the contrast between the color of the object and the background, the more readily will an object of any given size be distinguished. Thus, it is frequently observed that a visual acuteness of 6/9, with diminished illumination, is raised to 6/6 with a maximum illumination, as a result of heightened contrast between the type and its background. In cases of ametropia and amblyopia it is, however, also frequent that increased illumination reduces the definition, owing to a superabundance of extraneous light, which serves to reduce the contrast within the polar field of fixation. In optical instruments it is found practicable to exclude peripheral extraneous light by means of a diaphragm of suitable aperture, and it is even possible to increase the definition, through limiting the field in an inferior instrument by further reducing the size of this aperture. Thus it is that the pin-hole disk heightens the visual acuteness in ametropes who view distant objects through it, and even though the same proportionate improvement can be obtained in a similar

[illegible]

[illegible] upper and lower [illegible] proportioned that [illegible] optically [illegible] Its conception is [illegible] contained two lenses [illegible] above, the other below their [illegible] straight lined edges within the frame. [illegible] a subsequent improvement in which [illegible] integral parts of a single lens, one of whose [illegible] two intersecting spheric curvatures of differ- [illegible] the required difference in power. In the higher [illegible] inefficient on account of its prismatic action and

[illegible] It rests on sound physiologic principles, [illegible] H. Knapp, New York, 1897.

the limited area of its upper field, wherefore, it has been supplanted by the so-called cemented bifocal lens invented by Morck, Fig. 71, in which the amplified power of its lower field is secured through attachment to the major lens, by balsam, of a smaller lens or segment of any desirable size or contour. When the segment is round and ground to a "knife-edge" it is known as the *Opifex* bifocal lens. By this means it is possible to place the optic center of the upper or lower lens in line with the visual axis of the eye when either field is used. See section on *Applications of the Prism-dioptry*. However, recently a solid bifocal lens has been exploited in which the lower field is obtained through intersection of two concave surfaces on the concave side of a meniscus, thereby producing an upper field that is greater than the lower field and generally obviating the prismatic action so manifest in the original, solid bifocal lens.

The patentees furnish so-called blanks, in which the segmental reading portion in the concave side of the blank is weaker, by quarter-dioptry intervals, between 1 and 3.5 dioptries, than the distance portion of the blank, which is given either a — 4 D. or — 6 D. surface. For instance, through grinding the opposite side of the blank to a convex spheric or toric surface, the residual positive power in the lower field is greater than the upper field in the resultant lens, which is a deep meniscus, this being the only form in which the blanks are now being supplied. Still, other unrevealed methods have lately been employed to produce equally efficient solid bifocal lenses. In Borsch's fused bifocal lens the lower field is obtained through fusing a small lens of flint-glass into a corresponding cavity placed eccentrically in the major lens of crown-glass, and this lens is subsequently finished so that the surfaces of both the upper and lower fields constitute a uniformly continuous surface, thereby obviating the dust-collecting raised edge of the segment common to ordinary cemented bifocal lenses, while also making the integral segment almost invisible. Borsch's fused lens is now commercially known as the Kryptok lens, which probably derives its name from the hidden recess or crypt into which the flint-glass lens is fused. Flint-glass, on account of its higher refractive index, is here used to produce increased power, and not to effect *achromatism* (freedom from color), which, in fact, is not a possibility under the prescribed conditions, since the unequal dispersive powers of the crown-glass and flint-glass and the proportions to each other of their surface-curvatures, as required in achromatism, are not taken into consideration. See section on the *Fundamentals of Achromatism*.

In order that a lens shall be achromatic, its component superposed

contra-generic lenses of crown-glass and flint-glass, of unequal refractive powers and dispersions, must have their curvatures so accurately proportioned to each other that the chromatic (color) aberration produced by one lens is counteracted by that of the other. In fact, it is the same principle that applies to the achromatic prism, wherein the two component prisms of unequal refractions and dispersions are placed with their bases in reversed positions, and their angles are so relatively proportioned as to still effect ray-deviation without any appreciable amount of dispersion or decomposition of white light into its spectral colors. Consequently, fused bifocal lenses are not designed to be achromatic, since their construction is directly opposite to this requirement, so that they are frequently even less free from perceptible color than ordinary cemented bifocal lenses of crown-glass.

Although the various types of ophthalmic lenses are not generally made to insure achromatism, its fundamental principle as well as the color-phenomenon so frequently observed in them are nevertheless thought to merit at least the following elementary description in an essay of this kind.

The Fundamentals of Achromatism.

Dispersion.*

In the year 1672 Sir Isaac Newton communicated to the Royal Society his discovery that refraction by a prism resolves white light into its component elementary color-rays to produce the solar spectrum, and that these dispersed rays of different wave-lengths may be again refracted by a suitably placed duplicate prism to produce white light. Although Newton thus laid the foundation for achromatism, and incidentally also substantiated the law that light of any wave-length is propagated either forward or backward over the same rectilinear course, he was nevertheless of the opinion that the color-deviation could not be corrected in a prism or lens without annulment of the ray-deviation. Seventy-five years later Euler, who had based his argument upon the erroneous assumption that the human eye is an achromatic combination of lenses, mathematically demonstrated that achromatism is achievable. Ten years thereafter Dolland, though having failed to experimentally verify Euler's calculations, yet encouraged by the example of Klingenstierna's experiments, successfully produced an achromatic telescope-objective in which a convex

* This introduction to achromatism is abstracted chiefly from Prof. Southall's *Principles of Geometrical Optics.*

lens of crown-glass was combined with a concave lens of flint-glass; the chromatic dispersion of the former being counteracted by the latter through an equal amount of chromatic dispersion in the opposite direction.

Newton had concluded that sun-light is not homogeneous, but is composed of rays of different colors, some of which are more refrangible than others; the red rays being refracted least, and the violet rays most, in the spectrum whose colors are arranged in the order of red, orange, yellow, green, blue, indigo and violet. Moreover, he supposed the solar spectrum to be continuous with respect to the gradual transmutation of its colors. However, Wollaston, in 1802, discovered that the solar spectrum is crossed by a number of dark bands, which were later explained by Kirchhoff to be due to the light-waves from each incandescent element within the nucleus of the sun being absorbed by its own cooler gas within the sun's vaporous envelop. In fact, the absence of any particular color or light-wave in the solar spectrum, as exhibited by the presence of these bands, indicates that the corresponding wave is either not emitted by the sun, or else that it is lost through absorption or otherwise before it reaches the observer's eye.

After all, it was not until Fraunhofer, in 1814, independently rediscovered these bands as lines irregularly distributed over the entire extent of the solar spectrum, and that only their relative distances apart and their associated color-areas are altered in the same sequence by using a prism of different material, that their true significance and value were recognized by him. As Fraunhofer found each of these lines, at that time 576 in number, to correspond to a definite wave-length of light, he employed them to determine the different refractive indices of a substance, and designated the more conspicuous lines in the different parts of the spectrum by the capital letters of the Roman alphabet from A to H; the line in the violet end of the spectrum, as nearly as he could locate it, being designated by the letter J. Therefore, it is obvious that the so-called refractive index of a substance is ambiguous, unless the particular line of the spectrum for which the refractive index has been determined is specified, since a medium has just as many indices of refraction as there are lines in the different color-areas of the spectrum. Hence, the line, D, located in the brightest yellow part of the spectrum, is commonly used to determine the refractive indices of the various kinds of glass, so that the refractive index for this line, indicated by n_D, identifies the so-called *mean ray*, situated about midway between the lines, C and F, in their respective red and blue parts of the spectrum. As the interval

between these lines defines the chosen extent of the dispersions of two media that may become subjects of comparison, the difference between them, $n_F - n_C$, called the *mean dispersion,* serves for all practical purposes to sufficiently characterize the dispersions of different substances; and the ratio of this value to the difference, $n_D - 1$ (the index for the line D, less the index of air, whose value is 1), provides an appropriate expression for the so-called *relative dispersion* or *dispersive power,* $\frac{n_F - n_C}{n_D - 1}$*, of a substance. This value, when applied to a definite material, such as crown-glass, may be written, $\frac{n_F' - n_C'}{n_D' - 1}$, so as to distinguish it from the corresponding value, $\frac{n_F'' - n_C''}{n_D'' - 1}$, for flint-glass.

Achromatism, as will be later shown, depends (1) upon the joining together of at least two kinds of glass of different refractive powers and unequal dispersions, as identified by the lines of the spectrum, and (2) upon a definite choice of the inclination of their contiguous surfaces to their exposed bounding surfaces, since the latter are usually prescribed. Although Fraunhofer and others had tried to produce new kinds of glass, it was not until Abbe and Schott, of Jena, Germany, succeeded in making a variety of pairs of different kinds of crown-glass and flint-glass, in which the dispersion in the various parts of the spectrum is as nearly as possible proportional in each pair, that it has been possible to take the fullest advantage of Fraunhofer's discovery. Encouraged by these experiments, Schott and Genossen, in 1884, established the "Glasstechnisches Laboratorium" at Jena, where the world renowned Jena glass is produced to supply skilled opticians with glass having the properties necessary to fulfil, as closely as possible, the theoretic requirements in perfected optical instruments.

* The reciprocal of this value, which is $\frac{n_d - 1}{n_f - n_c}$, is designated [illegible] differences between the refractive indices of n_d, or n_f and [illegible] spectral lines are given under "Partial Dispersions" in the [illegible] of Jena glass.—*Principles of Geometrical Optics,* Southall.

Achromatism.

In order to explain the principle of achromatism, license is here taken to depart from the usual method, by substituting an original, analytic and more or less elementary one, in which, for the sake of greater clearness in the accompanying drawings, the delineations of dispersion are greatly exaggerated. Moreover, the Fraunhofer D-line in the brightest yellow part of the spectrum is chosen for the refracted mean ray (Y), about midway between the red (R) and blue (B) rays; wherefore this demonstration also strictly applies only to the behavior of the color-rays confined between these limitations of the spectrum, commonly designated by the Fraunhofer lines C and F, respectively, for incident white light proceeding from infinity. As the indigo and violet rays, though necessarily constituent parts of white light, are thus ignored, it is to be understood that, wherever emergent white light is later mentioned it is supposed to be only approximately colorless.

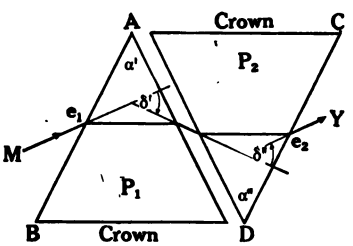


Fig. 116.

In geometric optics the laws of refraction are ordinarily applied to white light as if it were homogeneous or monochromatic and traversed successive media as though each medium had only one index of refraction, usually the index, n_D, corresponding to the D-line of the spectrum.

Thus, for instance, when the axes (mean rays) of the incident and emergent beams are inclined at equal angles to the surfaces of a prism, P_1, in Fig. 116, the prism is said to be in position to produce the "minimum deviation" of the emergent beam, as expressed by

$$\delta' = (n'_D - 1)\alpha', \quad \text{XVII.}$$

regardless of the fact that it is composed of a number of constituent color-rays. Therefore, strictly speaking the prism can not be placed in position [illegible] all color-rays at the same time, bu[illegible] [illegible]tween the red and blue [illegible] a prism of very small

Since achromatism involves the deviations as well as the dispersions produced by at least two media of different optical density, only the deviation of a monochromatic ray, the mean yellow ray, will first be considered with reference to two adjacent prisms, P_1 and P_2, Fig. 116, of the same crown-glass. In this case the incident mean ray, M, to the prism, P_1, suffers the minimum deviation, δ', in its progress toward the second prism, P_2, where it again suffers the minimum

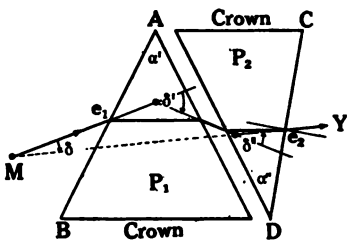


Fig. 117.

deviation, $\delta'' = \delta'$, in its passage through the second prism, since both prisms have the same refracting angle, $\alpha' = \alpha''$. As these equal deviations occur in opposite directions, there is not any actual deviation from the direction of the incident ray, M, so that the emergent ray, Y, is parallel to it, and the prisms jointly act as a plate, even though their faces are not in contact.

However, deviation from the direction of incidence may be secured

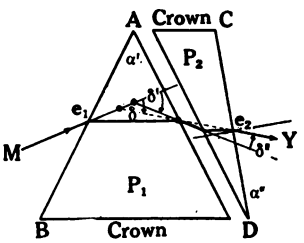


Fig. 118.

through changing the inclination of the posterior surface of the second prism; in other words, through decreasing its refracting angle, α'', as shown in Fig. 117, where the emergent ray, Y, has a lesser upward direction than in Fig. 116. In Fig. 118 the angle, α'', is even still smaller, and therefore produces a downward direction of the emergent ray Y.

It is obvious that these changes in the direction of the emergent ray, Y, depend upon whether the refracted ray within the second

prism approaches the point, e_2, of emergence from below or from above the perpendicular at this point of the surface, CD, as shown in figures 117 and 118, this also being true of the same mean ray, if considered as the axis of the dispersed color-rays, when the incident light is a beam of white light. In any event, the resultant deviation is $\delta' - \delta'' = \delta$; this being the same deviation as would be produced

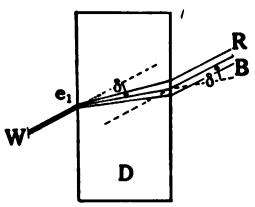


Fig. 119.

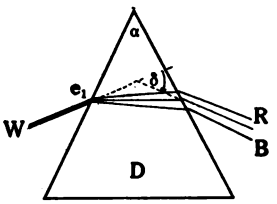


Fig. 120.

by a single prism whose angle between the faces, AB and CD, is $a = a' - a''$.

However, in associating deviation with dispersion it is necessary to consider also the individual directions of the color-rays. Thus, for instance, in the figures 119 and 120, representing the principal sections of a plate and a prism, respectively, the narrow beam of white light, W, incident at e_1, upon the denser medium, D, is refracted by it so as to produce a slightly wider beam composed of emergent

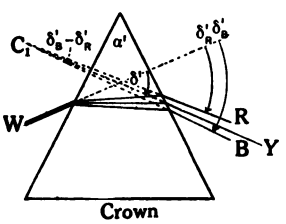


Fig. 121.

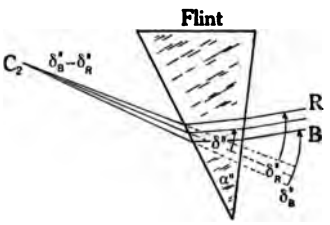


Fig. 122.

color-rays between red (R) and blue (B). However, to render this color-effect at the margins of the emergent light from a plate at all visible, even through a carefully adjusted narrow opening, the plate would require to be very thick, and the incident light would also have to pass through a very narrow slit; but, under normal conditions of exposure, the incident light may be thought to consist of an infinite number of contiguous beams, W, whose constituent and emergent parallel color-rays so overlap each other as to be perceived by the eye as white light. Still, in each of the indicated media the color-rays are divergent at e_1, when the light is propagated from left to right,

but, in accordance with the law of reversibility, when the color-rays between R and B are propagated in the opposite direction, they also internally converge toward e_1, and there produce emergent white light. Hence, in either case, white light is manifest whenever its constituent color-rays are at least parallel; whereas, an actual evidence of color is confined to the absence of such parallelism after refraction.

To the degree of approximation previously stated, the prism, Fig. 121, is placed in position of minimum deviation, so that the angular deviation of the refracted blue ray, B, from the direction of incidence is $\delta'_B = (n'_B - 1)a'$. Similarly, for the red ray, R, the deviation is $\delta'_R = (n'_R - 1)a'$.

When a' is herein substituted by $\dfrac{\delta'}{n'_D - 1}$, from the equation XVII, the angular difference between the blue and red rays is

$$\delta'_B - \delta'_R = \frac{n'_B - 1}{n'_D - 1}\delta' - \frac{n'_R - 1}{n'_D - 1}\delta' = \frac{n'_B - n'_R}{n'_D - 1}\delta',$$

or, the angle of dispersion, $\delta'_B - \delta'_R$, is equal to the dispersive power $\dfrac{n'_B - n'_R}{n'_D - 1}$, multiplied by the angular deviation, δ', of the refracted mean ray, Y. Therefore, not only is deviation and dispersion produced, but experience also teaches that different substances produce different dispersions. For instance, flint-glass has a greater dispersive power than crown-glass, so that an inverted flint prism may be found to produce the same, though oppositely directed, dispersion as a crown prism of greater angle. This is made clear through the reversed arrows ascribed to the rays emerging from the flint prism, of lesser angle, Fig. 122, which also shows the angular dispersion, $\delta''_B - \delta''_R$, for the color-rays *diverging* from C_2 in their incidence from the left, since the refracted parallel rays R and B, if propagated from the opposite direction, actually incur the same dispersions, δ''_R and δ''_B, respectively.

Therefore, it is obvious that two prisms having different angles of *deviation*, δ' and δ'', respectively for crown-glass and for flint-glass, yet producing the same angles of dispersion, $\delta'_B - \delta'_R = \delta''_B - \delta''_R$, may be combined with their bases reversed so as to insure the emergent color rays being made parallel to constitute white light, and that

the choice of their refracting angles will require to be such that, the divergent color-rays, emergent from the crown-glass prism, shall be those rays, incident to the flint-glass prism, which appear also to proceed from the common point C_1, the vertex of the coincident equal angles of dispersion in Fig. 123 and Fig. 124.

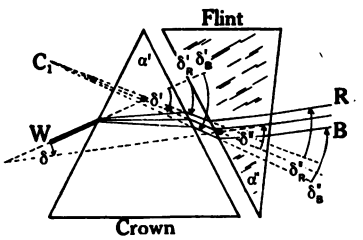


Fig. 123.

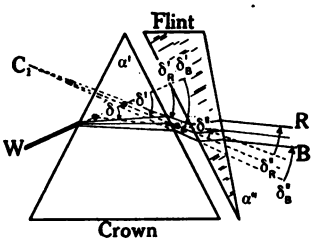


Fig. 124.

Since this principle also applies when the space between the parallel faces of the prisms is eliminated, it is usual to join them together by means of Canada balsam, thus creating a composite prism which is said to be achromatic only for the two colors red and blue. As the direction of each color-ray depends upon its own refractive index for each different medium, it follows that the angle of dispersion, $\delta'_B - \delta'_R = (n'_B - n'_R)a'$, for the crown prism, and similarly, $\delta''_B - \delta''_R = (n''_B - n''_R)a''$, for the flint prism.

Moreover, as these angles of dispersion must be equal to insure achromatism for the red and blue rays, it follows that:

$$(n'_B - n'_R)a' = (n''_B - n''_R)a'', \quad \text{.....XVIII.}$$

which determines the angles a' and a'' of the superposed prisms, in order that the emergent light shall be approximately free from color.

The deviation of the emergent light from the direction of incidence is equal to $\delta' - \delta'' = (n'_D - 1)a' - (n''_D - 1)a''$.

Thus, by making $\delta' = \delta''$, the angles of two prisms of different substances may also be so chosen as to avoid deviation of the refracted

mean ray, while still permitting the dispersion of all other color-rays, and as required in the construction of direct vision spectroscopes.

Incidentally it is to be noted that, the position of minimum deviation is not also the position of minimum dispersion, since the dispersion may be indefinitely increased by adjusting the prism with respect to the incident beam. Therefore, it is always possible to adjust the refracting angles, a' and a'' of two contiguous prisms of different glass so that the dispersion for two colors produced by the first prism may be sufficiently counteracted by the second prism to effect approximate freedom from color in the emergent beam. In this case, the residual dispersion, called the *secondary spectrum,* is due to the fact that different substances disperse corresponding parts of the spectrum to widely different extents. For instance, one medium may separate the blue and the orange light very much, while the other medium separates these colors very little, so that the areas of color in both spectra are not precisely alike. This so-called *irrationality of dispersion* prevents reliable comparisons of the spectra of prisms being made, since the spectrum obtained with one prism is not the same as that produced by another. Besides, the projected spectrum produced by a prism is impure, because each point in the spectrum is not illuminated exclusively by one of the constituent colors of white light, so that there is more or less overlapping of them. However, if the virtual spectrum is viewed through a prism in position for minimum deviation, there is not any sensible overlapping of this kind, hence the *virtual* spectrum is pure. On the other hand, spectra which are produced by *reflection* from ruled gratings are all pure and exactly alike, so that the data secured by different observers are always the same, and for which reason the diffraction spectrum is chosen as the standard in the physical laboratory.

However, reverting to the *refracted* spectrum, it is possible in an achromatic combination of two prisms to correct its successive residual colors through adding another prism for each color. When three prisms are used to correct three colors the remaining smaller dispersion is called the *tertiary spectrum.* Therefore, it is impossible to obtain perfect achromatism without the use of a very large number of different media, but, as the successive uncorrected spectra rapidly grow fainter they are negligible. Hence it is seldom deemed necessary to correct more than two colors.

The preceding principles are also well adapted to explain achromatism of a lens, especially when the component lens-thicknesses are exaggerated for the purpose only of lucid illustration.

With this understanding, all parallel rays of white light incident from the left upon the convex lens, Fig. 125, result in divergent pencils of color-rays within the lens, which upon emergence intersect the optic axis at different points, so that the red and blue rays are separately focused at R_1 and B_1, respectively. In other words, the convex lens is more strongly convergent toward the axis for blue rays than for red

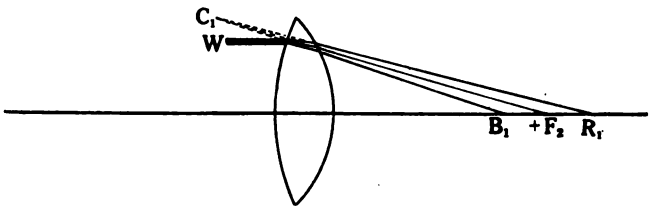


Fig. 125.

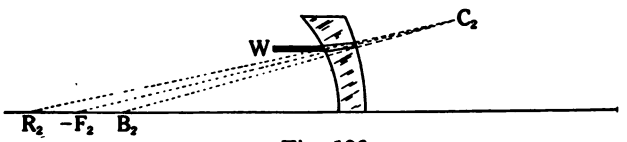


Fig. 126.

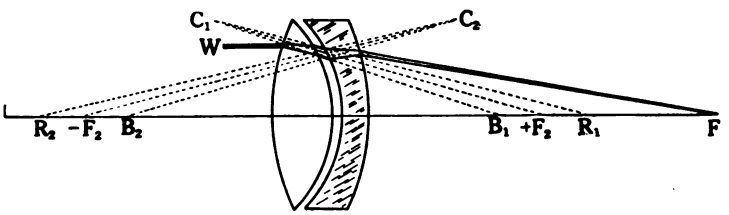


Fig. 127.

rays, or the emergent light is a divergent pencil of color-rays appearing to proceed from C_1; whereas, the weaker concave lens, Fig. 126, is more strongly divergent for blue rays than for red rays, so that the emergent light is a convergent pencil of color-rays whose vertex is C_2. Now, as the dispersion of flint-glass is greater than that of crown-glass, it is obvious that the same amount of dispersion, though in the opposite direction, may be produced through choice of a weaker concave flint lens whose focal length is, therefore, greater than that of the convex crown lens.

Hence, when two such properly proportioned lenses, Fig. 127, are very slightly separated upon a common axis, their *equal dispersions* in opposite directions counteract each other, thus producing an achromatic combination corrected for two colors. However, in this case the rays emitted from the convex lens are *incident* upon the

concave lens as a divergent pencil of color-rays which individually converge toward R_1 and B_1 upon the axis, so that the divergent color-rays from the convex lens, which appear to proceed from C_1, are no longer, after refraction by the concave lens, directed to the point C_2, but assume a lesser convergence toward the more remote point, F, the focus of the combination. Therefore, in order to make a lens approximately achromatic for parallel incident rays it is necessary that it shall be so constructed as to effect a fusion of the points B_1 and R_1 of the convex lens into a single point, F, the positive focus of the combination. This means that the emergent color-rays are not parallel, and, therefore, do not constitute white light until they have collectively merged into the focal point, F, where they are superposed to produce it; whereas, with two prisms the emergent light is achromatised through immediate parallelism of the color-rays at the surface of emergence, and in front of which a continued divergence of the internal color-rays has been shown to exist. On the other hand, in a combination of two contra-generic lenses continued divergence of the color-rays within both lenses does not exist, since their divergence within the crown lens is approximately counteracted by their convergence within the adjoining flint lens, Fig. 127. Consequently, it is evident that white light does not traverse the interior of a refracting medium, even if it is achromatic. Since the convex crown lens has a shorter focal length, it dominates the refractive power; whereas, the weaker concave flint lens prevails only as the color-correcting element in the combination, which in practice is made integrate through Canada balsam, in order to avoid reflection from the contiguous surfaces.

Moreover, as the angles of dispersion are proportionate to their corresponding relative dispersions and may be expressed through them, it is also obvious that the relative dispersion of the convex *crown* lens must bear the same proportion to its mean focal length, $+f'$, as the relative dispersion of the concave *flint* lens to its mean focal length, $-f''$; that is to say,

$$\frac{n'_F - n'_C}{n'_D - 1} : f' = \frac{n''_F - n''_C}{n''_D - 1} : -f''.$$

This equation, when transformed to

$$\frac{n'_F - n'_C}{n'_D - 1}\,\frac{1}{f'} + \frac{n''_F - n''_C}{n''_D - 1}\,\frac{1}{f''} = 0, \quad \ldots\ldots\ldots\ldots \text{XIX.}$$

prescribes the condition of achromatism of two superposed contra-generic thin lenses made of crown-glass and flint-glass, and involves the stipulation that the sum of the products of the relative dispersions of the substances used and the refractive powers of the lenses shall be zero.

As n'_F is greater than n'_C, and n''_F is greater than n''_C, while n'_D and n''_D are both greater than unity, it follows that f' and f'' must have opposite signs, so that, if the combination is to be a positive or convergent lens, the component convex lens, whose focal length is f', will require to be stronger than the concave lens whose greater focal length is f''; wherefore, the relative dispersion of the convex material will have to be less than that of the concave material, in order to satisfy the equation XIX. This equation also applies to a *concave* achromatic lens, but in which the stronger concave element is made of crown-glass and the other of flint-glass.

However, as the achromatic combination under discussion is convex, its refraction is $1/f$, while the refractions of its component convex and concave lenses are $1/f'$ and $-1/f''$, respectively, so that,

$\frac{1}{f'} - \frac{1}{f''} = \frac{1}{f}$, from the general formula,

$$\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{f} \quad \text{.....XX.}$$

Therefore, so long as the focal distances, f' and f'', which satisfy this equation, are also so proportioned to each other as to satisfy the equation, XIX, the convex lens will be achromatic with respect to the two colors, red and blue.

The exposed surface-curvatures of a lens of any form, whose focal length is f, are determined by the general formula,

$$\frac{1}{f} = (n_D - 1)\left(\frac{1}{r_1} - \frac{1}{r_2}\right), \quad \text{.....XXI.}$$

so that even the achromatic lens may be of any type commensurate with the more or less empiric choice of the radii, r_1 and r_2, if it is to satisfy only the condition of achromatism for paraxial rays. But, as the image-distortion due to spheric aberration is nearly as pernicious as chromatic aberration, the former must also be corrected, especially in lenses of wide aperture.

A lens in which both the chromatic and spheric aberrations are cor-

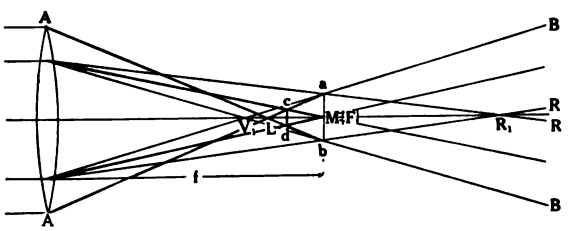
rected is called an *aplanatic lens,* yet its construction also involves compliance with the so-called sine condition (*Principles of Geometrical Optics,* Southall). However, ordinarily the radii, r_1 and r_2, of the outer surfaces of an archromatic lens are selected so as to give the lens that form which produces the least spheric aberration. In order to simplify the cumbersome calculations necessary to provide the requirements to avoid both kinds of aberration in thin lenses, Herschel, in 1821, established a simple rule, which, with sufficient practical accuracy, applies to lenses of no greater aperture than 1½ inches. Professor Prechtl of Vienna, in 1828, stated it as follows:

"An object-glass consisting of two thin lenses is practically free from aberration when the radius of the exposed surface of the crown-glass = 6.72 and that of the flint-glass = 14.2, provided the mean focal length, f, of the objective is made = 10, and the radii of the two contiguous inner surfaces are so calculated that the focal lengths of the component lenses are in proportion to their dispersive powers." However, this applies only in case the crown-glass index is 1.528 and the flint-glass index is 1.601, when it is merely necessary to make the radius of the exposed surface of the crown-glass, $r_1 = 0.672 \times f$, and the radius of the exposed surface of the flint-glass, $r_2 = 14.2 \times f$, in order, through the formula, XXI, to determine the radii of the contiguous surfaces of the lenses whose focal lengths, f′ and f″, must be made to satisfy the equations, XIX and XX, for an achromatic lens whose focal length is f. Moreover, even though Herschel's rule to obviate spheric aberration is not complied with, the formulæ, XIX, XX and XXI, provide for at least approximate achromatism of a lens of any form for which at least one radius of surface-curvature and the focal length are known. However, when achromatic lenses are required to be of wide aperture, it is necessary to use formulæ of a higher order of approximation.

Through superposing the diagrams illustrating spheric and chromatic aberration, figures 107 and 108, to produce the Fig. 128, the distinction between the achromatic and the aplanatic lens may be lucidly shown, since all of the color-rays of the simple lens that intersect the optic axis, between V_1 and R_1, are made to merge into the focal point, M, by the achromatic lens, which, if also aplanatic, eliminates both the longitudinal spheric aberration, LM, and the chromatic aberration, $V_1 R_1$.

Incidentally it is also to be noted that the detrimental effect of chromatic aberration is greater than that of spheric aberration, because the circle of chromatic aberration, ab, is considerably greater than the least circle of spheric aberration, cd.

However, as in the latter the light intensity is correspondingly greater, its effect is not to be estimated by the proportions of these circles of aberration to each other. The diagram makes it apparent that the *longitudinal chromatic aberration,* $V_1 R_1$, will increase for a greater focal length, f, of a lens made of the same kind of glass, and that it does not depend upon the aperture of the lens, but that it will remain the same constant for different apertures of a lens having the same focal length. Therefore, it is the reverse of the condition that applies to the *longitudinal spheric aberration,* LM, which increases with the square of the aperture, AA, and in inverse proportion to the focal distance. On the other hand, the *lateral chromatic aberration,* measured by the diameter, ab, of the circle of chromatic aberration, depends upon the diameter of the aperture of the lens, and is proportional to it, as may be seen in the diagram. Therefore, in an



Circle of Chromatic Aberration, ab, in Focal Plane.
Least Circle of Spheric Aberration, cd.

Fig. 128.

achromatic convex lens it is evident that the weaker concave lens must supply a circle of chromatic aberration of the same size as that produced by the stronger convex lens; wherefore, the dispersive power of the concave lens must be greater in proportion to its refractive power, and for which reason, as previously shown, flint-glass having a greater dispersive power is used. Moreover, the greater the dispersive power of the flint-glass is in proportion to that of the crown-glass, and the smaller the difference between their respective dispersions, the flatter the curvature of the concave lens will require to be, and therefore also the greater will be the preponderance of the refraction of the convex lens over that of the concave lens.

Dioptric Formulæ for Combined Cylindric Lenses.

Shortly after publication of the author's "*Treatise on Ophthalmic Lenses,*" the late Dr. Swan M. Burnett kindly suggested the execution of plastic models of combined cylindric lenses, by placing a set of these, conceived and hastily prepared by himself, in the writer's hands

for further elaboration; with the request, if possible, also to produce two additional combinations in which the cylinders were to be united at angles other than right angles. As a result of the writer's research during the time devoted to the construction of the latter more especially, and with a view to establish confidence in the precision of these models, the following mathematical demonstration was first presented in 1888. The subject has been divided into two theorems; one covering congeneric cylinders, the other, contra-generic cylinders, and in which the diagrams are intentionally drawn at variance with the laws of true perspective, in order to strictly preserve all important circles and right angles referred to in the text. The final transformations of the formulæ, as adapted to the requirements of the *dioptral* system, are included so as to render complete the proof that two superposed cylindric lenses, crossed at any angle, are equivalent to a spherocylindric lens, thus actually nullifying the need for employment of the formulæ in ordinary ophthalmic practice. However, the importance of the demonstration is made evident in the *sixteen laws* originally disclosed by the author, through simple formulæ based upon *first approximation,* and which ordinarily apply to thin lenses that are commonly used to counteract optical anomalies of vision.

I. Congeneric Cylinders.

1. *Relative Positions of the Primary and Secondary Planes of Refraction.*

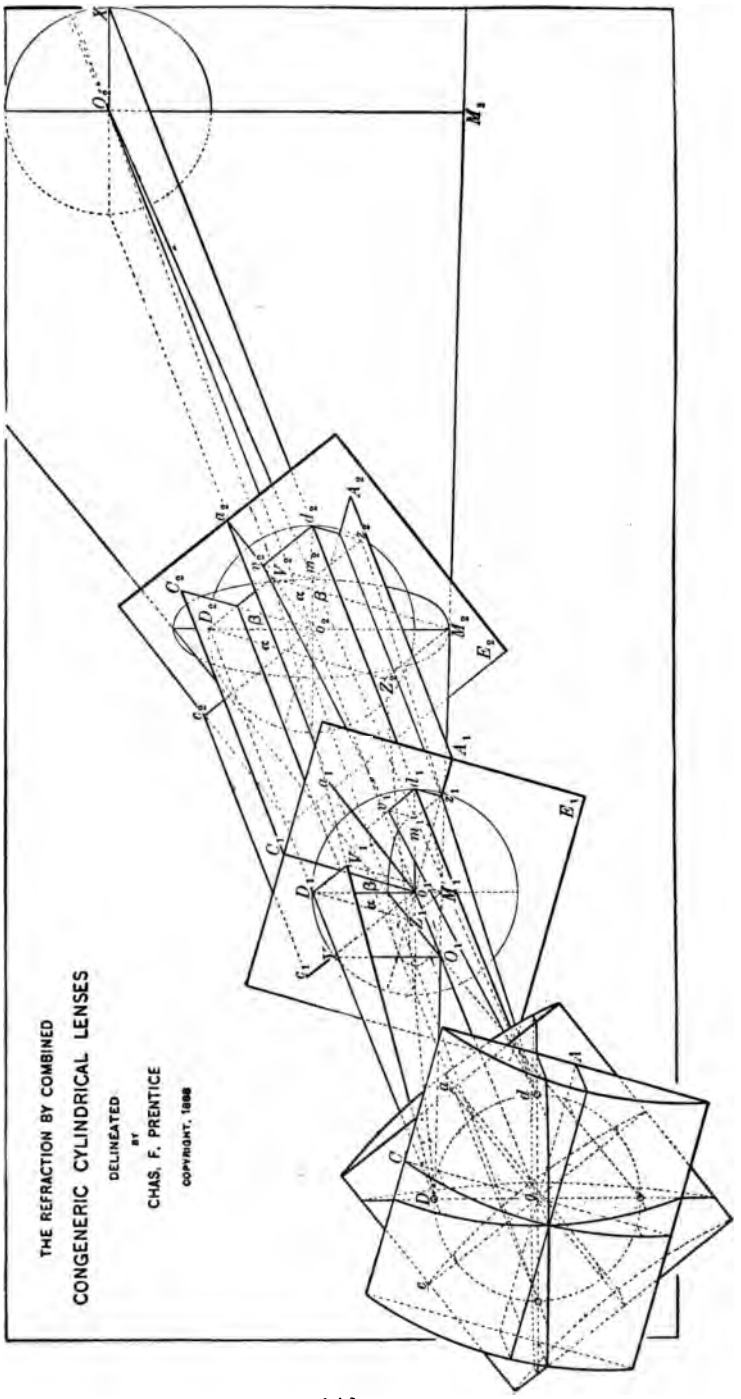
As in most cases the total thickness of the combined cylinders is a very small dimension compared with their resultant focal lengths, the cylinders may be considered to be so thin that their individual optic centers coincide in one point, within a plane perpendicular to the optic axis, this point being taken as the optic center of the superposed lenses.

In Plate I, two combined convex cylindric lenses are shown, which, though somewhat at variance with the prescribed conditions of thickness, will, however, better serve to make our subject clear.

The dotted circle shown within the lenses, with its center at the optic center, o, shall represent the plane above alluded to.

The *passive* or axial planes of the cylinders are shown by dotted parallelograms at A and a, bisecting each other under the angle $Aoa = \gamma$ in the optic axis at o; and their *active* planes of refraction, C and c, which are of necessity at right angles to their correlative axial planes, similarly bisect each other at the same point. Hence, $\sphericalangle Coc = \sphericalangle Aoa = \gamma$.

PLATE I


THE REFRACTION BY COMBINED
CONGENERIC CYLINDRICAL LENSES
DELINEATED
BY
CHAS. F. PRENTICE
COPYRIGHT, 1898

The compound lens thus presented consists of two congeneric cylindric elements, each of which, *independently considered,* will have its corresponding focal plane, which, for convenience, we may term an *elementary* focal plane of the combination. Thus, E_1 and E_2, at the focal distances, f_1 and f_2, are the elementary focal planes for the cylinders, C and c, respectively. The cylinder, C, will consequently have the property of deflecting a ray, incident at D, perpendicularly from D_1, in the plane E_1, to the point Z_1 of the axial plane, $A_1 Z_1$; whereas, the cylinder c will have the property of deflecting a ray, incident at the same point, perpendicularly from D_2, in the plane E_2, to the point, V_2, of the axial plane, a_2o_2.

The greatest amplitude of deflection for C will therefore be D_1Z_1 in the plane E_1, and for c will be D_2V_2 in the plane E_2. It is further manifest that the refracted ray, DV_1V_2, contributed by c only, in attaining its greatest deflection D_2V_2, in the plane E_2, will penetrate the plane E_1 at V_1, and in it present a proportionate deflection D_1V_1.

D_1Z_1 and D_1V_1, being amplitudes of deflection *reduced to the same plane,* E_1, will then bear the same relation to each other as their corresponding refractions. Thus

$$D_1Z_1 : \frac{1}{f_1} = D_1V_1 : \frac{1}{f_2};$$

or,

$$D_1Z_1 = \frac{1}{f_1}, \text{ when } D_1V_1 = \frac{1}{f_2},$$

and which may easily be shown to be the case when the deflections are measured in a plane one inch from the lens (*Refraction and Accommodation of the Eye,* by E. Landolt, M. D., Paris, translated by C. M. Culver, M. A., M. D., Philadelphia, 1886, see page 58).

Provided, therefore, that the deflections are measured, within the same plane, from a point D_1 of the same line of incidence, DD_1, we may determine the resultant of two deflections, D_1Z_1 and D_1V_1, for any angular deviation existing between them at D_1, by the physical law governing similarly united forces. D_1M_1, as the diagonal of the parallelogram, $D_1V_1M_1Z_1$, will consequently be the resultant deflection accruing from a combination of the cylinders, C and c.

As each cylinder contributes a plane of active and one of passive refraction, we shall evidently obtain two resultant principal planes for their combination, the one of greatest refraction, commonly called the *primary* plane, DD_1o_1o, intersecting the angle, $Coc = \gamma$, between

the active planes of refraction, C and c, and one of least refraction, termed the *secondary* plane, dd_2o_2o, intersecting the angle, $Aoa = \gamma$, between the passive or axial planes, A and a.

The primary plane, in penetrating the plane E_1, will consequently divide the angle $C_1o_1c_1 = Coc = \gamma$ into $D_1o_1c_1 = \alpha$ and $D_1o_1C_1 = \beta$. In the plane E_1 we shall then find the angles α and β to be directly dependent upon the associated deflections D_1Z_1 and D_1V_1 for the point D_1. In the plane E_2 a similar division of the angle, $A_2o_2a_2$, by the secondary plane, will be rendered dependent upon d_2v_2 and d_2z_2 for the point d_2. As to this, the diagram is believed to be sufficiently clear, without further reference.

Since the resultants D_1M_1 and d_2m_2 define the directions of the refracted rays DM_1 and dm_2, it is further evident that for D and d to be points of the primary and secondary planes, respectively, they will have to be so chosen that D_1M_1 and d_2m_2 shall be directed *towards* the optic axis oo_1o_2; and, as we shall later learn, this is but one of the restrictions which renders a diagram somewhat difficult of construction. The resultant deflections D_1M_1 and d_2m_2 are, therefore, shown in the primary plane, *coincident* with D_1o_1, and in the secondary plane *coincident* with d_2o_2, respectively.

For all intermediate points of the circle, the resultant deflections deviate *from* the optic axis. This has been taken advantage of in constructing Dr. Burnett's models, and in determining the directions of twelve refracted rays in each of the figures 2, Plates II and IV.

The position of the primary plane DD_1o_1o, shown as dividing the angle, $C_1o_1c_1 = \gamma$, so that

$$\gamma = \alpha + \beta, \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

will then be determined by fixing the relations existing between α and β.

In the plane E_1, from the triangle $D_1Z_1M_1$, we have

$$D_1Z_1 : Z_1M_1 = \sin \sphericalangle Z_1M_1D_1 : \sin \sphericalangle Z_1D_1M_1;$$

$$\sphericalangle Z_1M_1D_1 = \sphericalangle D_1o_1c_1 = \alpha,$$

by parallelism of Z_1M_1 and c_1o_1; and, for similar reasons,

$$\sphericalangle Z_1D_1M_1 = \sphericalangle D_1M_1V_1 = D_1o_1C_1 = \beta.$$

$$\therefore D_1Z_1 : Z_1M_1 = \sin \alpha : \sin \beta,$$

$$Z_1M_1 = D_1V_1.$$

$$\therefore D_1Z_1 : D_1V_1 = \sin \alpha : \sin \beta. \ldots\ldots\ldots\ldots\ldots (2)$$

In the oblique plane DD_2V_2 we find

$$D_1V_1 : D_2V_2 = DD_1 : DD_2;$$

or, as DD_1 and DD_2 are the focal distances, f_1 and f_2, of the cylinders, C and c, respectively,

$$D_1V_1 : D_2V_2 = f_1 : f_2. \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Multiplying the equations (2) and (3), we obtain

$$\frac{D_1Z_1}{D_2V_2} = \frac{\sin \alpha}{\sin \beta} \frac{f_1}{f_2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Since D_1o_1 is the radius of the circle indicated, we may, for convenience, ascribe to it the value 1. We shall then have

$$D_1Z_1 = \sin \sphericalangle D_1o_1Z_1,$$

$$\sphericalangle D_1o_1Z_1 = C_1o_1Z_1 - \sphericalangle D_1o_1C_1.$$

$$\therefore \sphericalangle D_1o_1Z_1 = 90° - \beta.$$

$$\therefore D_1Z_1 = \sin (90° - \beta) = \cos \beta. \quad \ldots\ldots\ldots\ldots (5)$$

In the plane E_2 we similarly find

$$D_2V_2 = \sin \sphericalangle D_2o_2V_2,$$

$$\sphericalangle D_2o_2V_2 = \sphericalangle V_2o_2c_2 - \sphericalangle D_2o_2c_2.$$

$$\therefore \sphericalangle D_2o_2V_2 = 90° - \alpha.$$

$$\therefore D_2V_2 = \sin (90° - \alpha) = \cos \alpha. \quad \ldots\ldots\ldots\ldots (6)$$

Substituting the values for D_1Z_1 and D_2V_2 from (5) and (6) in the equation (4), we obtain,

$$\frac{\cos \beta}{\cos \alpha} = \frac{\sin \alpha}{\sin \beta} \frac{f_1}{f_2};$$

or, by multiplying both members of equation by 2 and transposing,

$$2 \cos \beta \sin \beta = 2 \cos \alpha \sin \alpha \frac{f_1}{f_2}$$

$$\therefore \sin 2\beta = \sin 2\alpha \frac{f_1}{f_2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

The position of the secondary plane dd_2o_2o, shown as dividing the angle $A_2o_2a_2 = \gamma$ into $d_2o_2a_2 = \alpha$ and $d_2o_2A_2 = \beta$, *provided* d_2o_2 *is perpendicular to* D_2o_2, will be determined by similarly fixing the relations between α and β. Here it can also be shown that

$$d_2z_2 : d_2v_2 = \cos\alpha : \cos\beta. \quad (8)$$

$$d_1z_1 : d_2z_2 = f_1 : f_2. \quad (9)$$

$$d_1z_1 = \sin\beta. \quad (11)$$

$$d_2v_2 = \sin\alpha. \quad (12)$$

whereby, as before, $$\sin 2\beta = \sin 2\alpha \frac{f_1}{f_2}.$$

We therefore conclude that:

1. *The primary and secondary planes of refraction are at right angles to each other for any angular deviation of the axes of two combined congeneric cylindric lenses.*

In a further consideration of the relation (7), $\sin 2\beta = \sin 2\alpha \frac{f_1}{f_2}$, we observe that the sines of the angles 2α and 2β, which are each always less than 90°, merely differ by the co-efficient $\frac{f_1}{f_2}$.

If, therefore, $f_2 = f_1$, which is the case when the cylinders are of equal refraction, the $\sin 2\beta$ will be equal to the $\sin 2\alpha$, and which can only be the case when $\alpha = \beta$, or, as $\alpha + \beta = \gamma$, when $\alpha = \beta = \frac{\gamma}{2}$; hence:

2. *For combined congeneric cylinders of equal refraction, the primary plane equally divides the angle between the active planes of the cylinders, and the secondary plane similarly divides the angle between the axial planes of the cylinders.*

In case, however, $f_2 > f_1$, which is the case when the refraction of the cylinder C is greater than c, then $\sin 2\alpha > \sin 2\beta$, or, when $\alpha > \beta$, so that

3. *For combined congeneric cylinders of unequal refraction, the primary plane, in dividing the angle between the active planes of the*

cylinders, will be nearer to the active plane of the stronger cylinder, and the secondary plane consequently nearer to the axial plane of the same cylinder.

This is also demonstrated in the diagram.

As, for a combination of two cylinders, C and c, under given angular deviation of their axes, the only known quantities will be f_1, f_2 and γ, it will be necessary to express α and β in terms of f_1, f_2 and γ.

This is accomplished through the equations (1) and (7):

$$\gamma = \alpha + \beta$$

$$\sin 2\beta = \sin 2\alpha \frac{f_1}{f_2},$$

when, after proper substitution and reduction, we obtain:

$$\cos \alpha = \sqrt{\frac{1}{2} + \frac{1}{2} \frac{f_1 + f_2 \cos 2\gamma}{\sqrt{f_1^2 + 2f_1 f_2 \cos 2\gamma + f_2^2}}}. \quad \ldots\ldots\ldots (I)$$

It will be unnecessary to seek β in the same manner, since, through (1), we find $\beta = \gamma - \alpha$.

When reducing this formula, for any given value of γ, pursuant to reasons later given, it should be observed that $f_2 > f_1$, in which case α, within the angle γ, is to be counted from the axis of the *weaker* cylinder.

2. *Positions of the Primary and Secondary Focal Planes.*

As the plane DD_1o_1o is the primary plane, it follows that all parallel rays incident in it between D and o will, after refraction, intersect the optic axis oo_1 at some point, which will be a point of the primary focal line. Therefore the resultant ray DM_1M_2, in attaining its greatest deflection D_1M_1 in the elementary plane, E_1, will establish the position of the primary focal line, through its previous intersection of the optic axis, oo_1, at the point, O_1.

In the secondary plane, dd_1o_1o, for similar reasons O_2 will be a point of the secondary focal line, though this point of intersection of the final ray dm_1m_2 with the optic axis is more distant, in consequence of the inferior deflection d_2m_2 in the plane, E_2.

Similar resultant deflections, at opposite cardinal points of the circle

within the lens, define the directions of their corresponding refracted rays. These rays not only limit the major and minor axes of the ellipses shown in the planes, E_1 and E_2, but also determine the lengths of the focal lines at O_1 and O_2. Thus O_2M_3 represents one-half of the secondary focal line at O_2. The primary focal line, in the secondary plane, perpendicular to YO_1 at O_1, has been omitted, to avoid possible misinterpretation of more important points of reference in the diagram. All rays parallel to the optic axis, incident at intermediate points of the circle within the lens, will, upon refraction, intersect the planes E_1 and E_2 at correlative points of the ellipses drawn thereon.

The circle of least confusion, T, will lie between the planes E_1 and E_2. (See Plate II, Fig. 2.) Its position may be determined through a simple formula given by Prof. W. Steadman Aldis in his discussion of the focal interval resulting from rays obliquely incident upon a spherical lens. (*Elementary Treatise on Geometrical Optics,* W. S. Aldis, M. A., Cambridge, 1886, see page 39.)

Our object being to determine the distances of the primary and secondary focal lines, or planes, from the principal plane within the combined cylinders, we shall proceed as follows:

In the primary plane DD_1M_1, we have

$$DY : DD_1 = YO_1 : D_1M_1.$$

Substituting, $DY = O_1o = F_1$, as the primary focal distance;

$$DD_1 = f_1;$$

$$YO_1 = D_1o_1 = \text{radius} = 1.$$

$$\therefore F_1 = \frac{f_1}{D_1M_1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (26)$$

In the parallelogram $D_1V_1M_1Z_1$, the angle between the forces, D_1V_1 and D_1Z_1, being equal to $\sphericalangle C_1o_1c_1 = \gamma$, we have, as the resultant deflection,

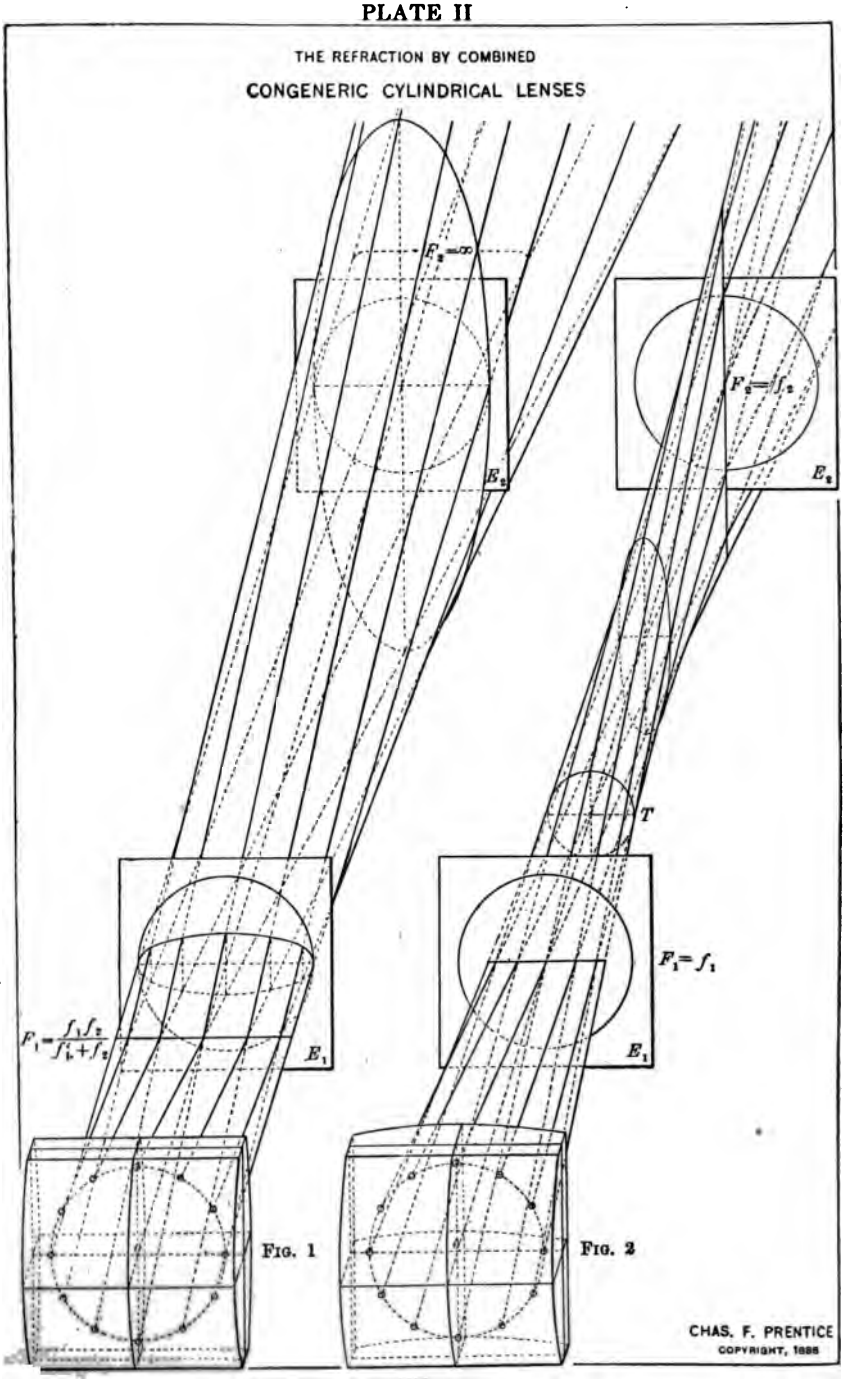
$$D_1M_1 = \sqrt{(D_1Z_1)^2 + (D_1V_1)^2 + 2(D_1V_1)\,(D_1Z_1)\cos\gamma}, \quad (27)$$

in conformity with the statical formula,

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\gamma},$$

for forces, P and Q, acting at the same point, within the same plane, under the angle γ.

PLATE II


THE REFRACTION BY COMBINED
CONGENERIC CYLINDRICAL LENSES
$F_2=\infty$
E_2
$F_2=f_2$
E_2
T
$F_1=f_1$
E_1
$F_1=\frac{f_1 f_2}{f_1+f_2}$
E_1
FIG. 1
FIG. 2
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Substituting in (27) the value of $D_1Z_1 = \cos\beta$, from (5); and of

$D_1V_1 = \frac{f_1}{f_2} D_2V_2$, from (3), $= \frac{f_1}{f_2} \cos a$, from (6), we obtain,

$$D_1M_1 = \sqrt{\cos^2\beta + \left(\frac{f_1}{f_2}\right)^2 \cos^2 a + 2\frac{f_1}{f_2}\cos a \cos\beta \cos\gamma}.$$

Introducing this value for D_1M_1 in (26),

$$F_1 = \frac{f_1}{\sqrt{\cos^2\beta + \left(\frac{f_1}{f_2}\right)^2 \cos^2 a + 2\frac{f_1}{f_2}\cos a \cos\beta \cos\gamma}} \quad \ldots (28)$$

By substituting the proper values for a and β, from equation (1), after adequate reduction we obtain

$$F_1 = \frac{f_1 f_2}{\sqrt{\frac{1}{2}\left[f_1(f_1+f_2\cos 2\gamma)+(f_1+f_2)\left(f_2+\sqrt{f_1^2+2f_1f_2\cos 2\gamma+f_2^2}\right)\right]}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (30)$$

Transforming, and substituting $1 - 2\sin^2\gamma$ for $\cos 2\gamma$, we may, for convenience in calculating, preferably write

$$F_1 = \frac{f_1 f_2}{\sqrt{\frac{(f_1+f_2)^2}{2} - f_1f_2\sin^2\gamma + (f_1+f_2)\sqrt{\frac{(f_1+f_2)^2}{4} - f_1f_2\sin^2\gamma}}}. \quad (II)$$

When the cylinders are of equal refraction, f_1 being equal to $f_2 = f$, the above assumes the simple form,

$$F_1 = \frac{f}{1+\cos\gamma} \quad \ldots\ldots\ldots\ldots\ldots\ldots (IV)$$

In the secondary plane dd_2XO_2, we have

$$dX : dd_2 = XO_2 : d_2m_2.$$

Substituting, $dX = O_2o = F_2$ as the secondary focal distance:

$$dd_2 = f_2;$$

$$XO_2 = \text{radius} = 1.$$

$$\therefore F_2 = \frac{f_2}{d_2m_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

In the parallelogram $d_2v_2m_2z_2$, the angle between the forces, d_2v_2 and d_2z_2, being equal to $\sphericalangle v_2d_2z_2 = 180° - \sphericalangle A_2o_2a_2 = 180° - \gamma$,

$$\therefore d_2m_2 = \sqrt{(d_2z_2)^2 + (d_2v_2)^2 + 2\,(d_2v_2)\,(d_2z_2)\cos(180° - \gamma)}.$$

Substituting the value for $d_2z_2 = \frac{f_2}{f_1} d_1z_1$, from (9), $= \frac{f_2}{f_1} \sin\beta$, from (11); and for $d_2v_2 = \sin\alpha$, from (12), we obtain,

$$d_2m_2 = \sqrt{\left(\frac{f_2}{f_1}\right)^2 \sin^2\beta + \sin^2\alpha - 2\frac{f_2}{f_1}\sin\alpha\sin\beta\cos\gamma};$$

which, introduced in (31) and being multiplied in the numerator and denominator by $\frac{f_1}{f_2}$, gives

$$F_2 = \frac{f_1}{\sqrt{\sin^2\beta + \left(\frac{f_1}{f_2}\right)^2 \sin^2\alpha - 2\frac{f_1}{f_2}\sin\alpha\sin\beta\cos\gamma}}.$$

Through substitution of the proper values for α and β, from equation (1), after suitable reduction we find

$$F_2 = \frac{f_1f_2}{\sqrt{\frac{1}{2}\left[f_1(f_1+f_2\cos 2\gamma)+(f_1+f_2)\left(f_2-\sqrt{f_1^2+2f_1f_2\cos 2\gamma+f_2^2}\right)\right]}} \quad \ldots\ldots\ldots\ldots (33)$$

Substituting, $\cos 2\gamma = 1 - \sin^2 \gamma$,

$$F_2 = \frac{f_1 f_2}{\sqrt{\frac{(f_1 + f_2)^2}{2} - f_1 f_2 \sin^2 \gamma} - (f_1 + f_2) \sqrt{\frac{(f_1 + f_2)^2}{4} - f_1 f_2 \sin^2 \gamma}} \quad \ldots\ldots\ldots\ldots (III)$$

This formula, reduced for cylinders of equal refraction, f_1 being equal to $f_2 = f$, becomes

$$F_2 = \frac{f}{1 - \cos \gamma} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (V)$$

It may be of interest to note that these formulæ differ from those given for F_1 merely by the minus sign in the denominator.

The preceding formulæ being alike applicable for combinations of convex or concave cylinders, the foci, f_1 and f_2, are to be introduced as positive values, merely with the restriction that f_2 be greater than or equal to f_1, in either case.

3. *Relations Between the Primary and Secondary Focal Planes.*

Since F_1 and F_2 have been shown to be dependent upon f_1, f_2, and γ, it is evident that, for fixed values of f_1 and f_2, the resultant foci will be rendered dependent entirely upon whatever value may be given to the angle, γ.

It is further obvious that the refraction of one cylinder will be affected most by the other when their axes coincide, or when $\gamma = 0°$, and least when their axes are at right angles to each other, or when $\gamma = 90°$.

We shall, consequently, fix upon the limits of F_1 and F_2 for these extremes of γ.

Introducing $\gamma = 0°$, and consequently $\cos 2\gamma = +1$, into the formulæ (30) and (33), we obtain, for $f_2 > f_1$,

$$F_1 = \frac{f_1 f_2}{\sqrt{\frac{1}{2}\left[f_1 (f_1 + f_2) + (f_1 + f_2)(f_2 + f_1 + f_2)\right]}} = \frac{f_1 f_2}{f_1 + f_2}.$$

$$F_2 = \frac{f_1 f_2}{\sqrt{\frac{1}{2}\left[f_1 (f_1 + f_2) + (f_1 + f_2)(f_2 - f_1 - f_2)\right]}} = \frac{f_1 f_2}{0} = \infty.^*$$

* The sign for infinity.

$$\therefore F_1 : F_2 = \frac{f_1 f_2}{f_1 + f_2} : \infty. \quad \ldots\ldots\ldots\ldots\ldots (34)$$

For $F_1 = \dfrac{f_1 f_2}{f_1 + f_2}$, we shall have as the refraction

$$\frac{1}{F_1} = \frac{1}{f_1} + \frac{1}{f_2}; \text{ consequently,}$$

4. *When the axes of the congeneric cylinders coincide, the primary focal plane will correspond to that focal plane which is defined by the sum of the refractions of the cylinders, whereas the secondary focal plane will be at infinity.*

This is shown in Plate II, Fig. 1.

Introducing $\gamma = 90°$, and consequently $\cos 2\gamma = \cos 180° = -1$, into (30) and (33), we have, for $f_2 > f_1$,

$$F_1 = \frac{f_1 f_2}{\frac{1}{2}\left[-f_1 (f_2 - f_1) + (f_1 + f_2)(f_2 + f_2 - f_1)\right]} = \frac{f_1 f_2}{f_2} = f_1.$$

$$F_2 = \frac{f_1 f_2}{\sqrt{\frac{1}{2}\left[-f_1 (f_2 - f_1) + (f_1 + f_2)(f_2 - f_2 + f_1)\right]}} = \frac{f_1 f_2}{f_1} = f_2.$$

$$\therefore F_1 : F_2 = f_1 : f_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (35)$$

As f_1 and f_2 correspond to the positions of the elementary planes, E_1 and E_2, it follows that

5. *The primary and secondary focal planes coincide with their correlative elementary focal planes, when the axes of the congeneric cylinders of unequal refraction are at right angles to each other.*

This is demonstrated in Plate II, Fig. 2.

In the same relation (35), if $f_1 = f_2$, then $F_1 = F_2$, or

6. *The primary, secondary, and elementary focal planes all merge into one plane, when the axes of the congeneric cylinders of equal refraction are at right angles to each other.*

As in this case we have but one focal plane, the refraction corresponds to that of a spherical lens.

F_1 being chosen to signify the primary focal distance, it will have to be less than F_2, yet if $f_1 > f_2$, we should find, in consequence of the relation (35), that $F_1 > F_2$. To retain the significances of F_1 and F_2, it will therefore be necessary to substitute f_2 by the greater given value of cylindric focus, and f_1 by the lesser, as stated under the formula (V).

Owing to the previous considerations, between the limits of 0° and 90° for γ, we are then to conclude that F_1 will vary between $\frac{f_1 f_2}{f_1 + f_2}$ and f_1, while F_2 varies between ∞ and f_2, as the nearest and most remote limits of focal distance for F_1 and F_2, respectively.

As an illustration, let Fig. 1, Plate II, represent two combined convex cylinders of unequal refraction, with their axes coincident, and so united as to permit of the rotation of one of the cylinders upon the true planes of their faces, about the optic center, o.

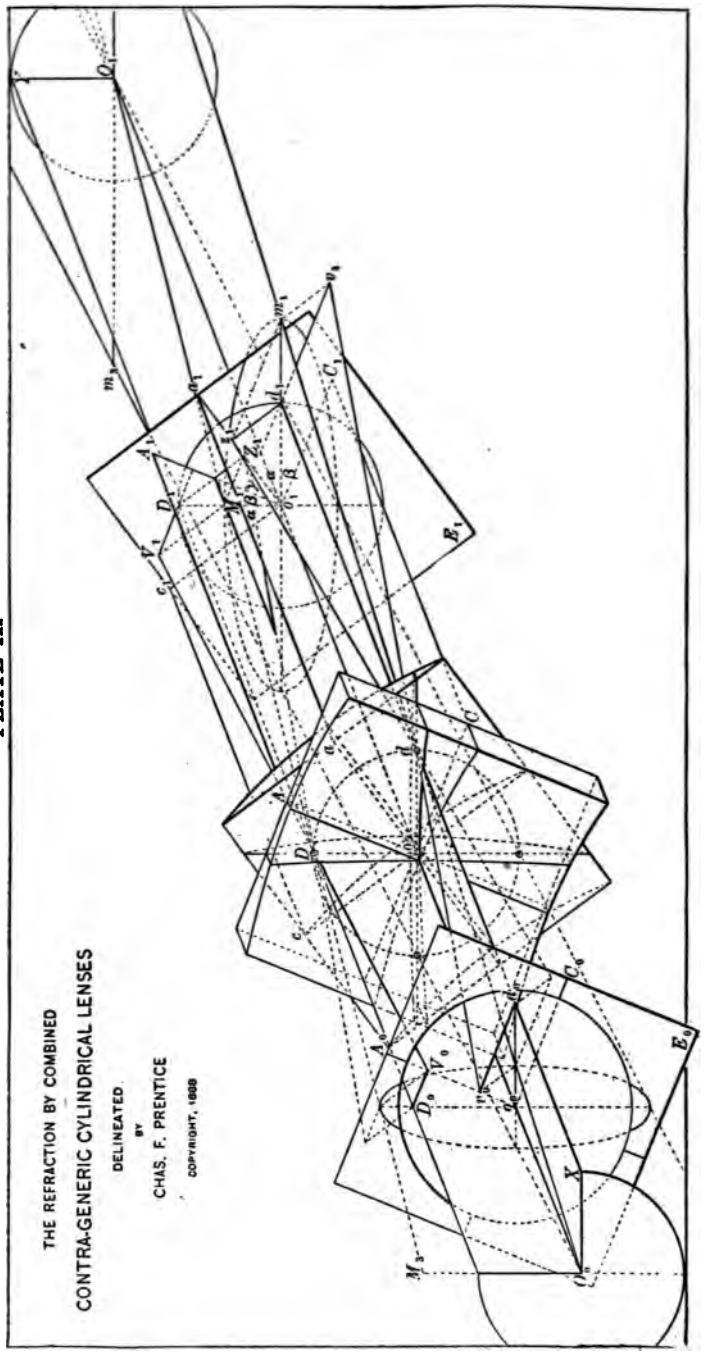
In the position shown ($\gamma = 0°$), the shortest possible focal distance F_1 of the primary focal line will be $\frac{f_1 f_2}{f_1 + f_2}$, which corresponds to the combined refraction, $\frac{1}{f_1} + \frac{1}{f_2}$, of the cylinders in the active plane. In the secondary plane, $F_2 = \infty$; consequently, $\frac{1}{F_2} = \frac{1}{\infty} = 0$, which corresponds to the refraction in the axial or passive plane of the cylinders.

The slightest change in the position of one of the cylindric axes will give rise to a definite value of the angle γ in the Formula III, thereby bringing F_2 within the limits of finite distance, while decreasing the value of F_1 in the Formula II.

For each successive increase in the angle γ, the primary focal plane corresponding to F_1, will recede farther and farther from the combined lenses towards E_1, while the secondary focal plane, corresponding to F_2, approaches nearer and nearer from ∞ to E_2, until $\gamma = 90°$, when F_1 will have reached E_1 on the moment that F_2 merges into E_2, as shown in Plate II, Fig. 2.

Rotation of one of the cylinders is thus associated with corresponding changes in the distances F_1 and F_2, while the movements of their

PLATE III


THE REFRACTION BY COMBINED
CONTRA-GENERIC CYLINDRICAL LENSES
DELINEATED
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correlative focal planes will be in opposite directions to each other; which proves that:

7. *The primary and secondary focal planes are conjugate planes, subject to variations of the angle between the axes of the congeneric cylinders.*

In order to comply with this law, in constructing the Plate I, it has been necessary to select elementary foci in marked disproportion to the curvatures of the cylinders; otherwise the secondary focus F_2 could not be brought within the space allotted for the diagram.

II. Contra-generic Cylinders.

1. *Relative Positions of the Principal Positive and Negative Planes of Refraction.*

In a combination of convex and concave cylinders, we can no longer have the primary and secondary planes, which we have learned to consider as planes of greatest and least refraction, but, instead, we shall have a plane of greatest positive and one of greatest negative refraction, synonymously with the generally-adopted distinction between convex and concave lenses, as designated by the signs + (plus) and — (minus), respectively. As the refractions of the convex and concave elements in the combination are opposing forces, the plane of greatest positive refraction will evidently lie between the active plane of the convex cylinder and the axial plane of the concave cylinder, whereas the plane of greatest negative refraction will be between the active plane of the concave cylinder and the axial plane of the convex cylinder.

In Plate III, therefore, the plane, DD_1o_1o, of greatest positive refraction is shown between c and A, and the plane, dd_1o_1o, of greatest negative refraction between C and a. These planes, being at right angles to each other, divide each of the angles, $A_1o_1c_1$ and $C_1o_1a_1$, into α and β.

To establish the formulæ for combined contra-generic cylinders, we shall, therefore, have to ascribe another significance to the angles α and β.

The deviation of the axes, Aa, is equal to angle $A_1o_1a_1 = \gamma$, and, since c_1o_1 is perpendicular to a_1o_1, $\alpha + \beta + \gamma$ is equal to 90°; consequently,

$$\alpha + \beta = 90° - \gamma \ldots\ldots\ldots\ldots\ldots\ldots (36)$$

The elementary focal planes, E_0 and E_1, corresponding to the focal distances f_0 and f_1, respectively, are exhibited on opposite sides of the

combined cylinders; since E_0, for the concave cylinder, is virtual and in the negative region before the lens; whereas, E_1, for the convex cylinder, is in the positive region behind the lens. Consequently, for the point D, the convex cylinder, c, contributes as its greatest amplitude of deflection D_1Z_1, perpendicular to a_1o_1 in the plane E_1. The greatest amplitude of deflection for the concave cylinder C is D_0V_0, perpendicular to A_0o_0 in the virtual plane E_0. As the incident ray at D will be refracted by the concave cylinder as if emanating from a correlative point V_0 of the virtual axial line V_0o_0, it is evident that the direction of the ray refracted by it will be V_0DV_1. The proportionate deflection contributed by the concave cylinder, measured in the plane E_1, will consequently be D_1V_1.

Provided the point D is properly chosen, it will be a point of the plane of greatest positive refraction, that is to say, when the resultant deflection D_1M_1, accruing from the associated deflections D_1V_1 and D_1Z_1 in the parallelogram of forces $D_1V_1M_1Z_1$, is directed *toward* the optic axis.

To insure D_1M_1 being so directed, it is obvious that the associated deflections, D_1Z_1 and D_1V_1, must also be measured in the plane E_1, in the positive region behind the lens.

Similar reasoning will apply to the point d as being in the plane, dd_1o_1o, of greatest negative refraction. In this instance d_1m_1, being a force directed *from* the optic axis, in the plane E_1, is to be taken negative, synonymously with the plane of greatest negative refraction.

The relations between α and β are to be determined by an analogous method to the one given for congeneric cylinders, whereby we obtain

$$\sin 2\alpha = \sin 2\beta \frac{f_1}{f_0}, \qquad (37)$$

as defining the positions of the planes of greatest positive and negative refraction, which are again at right angles to each other.

We here also find the sines of the angles, 2α and 2β, to differ by the co-efficient $\frac{f_1}{f_0}$. Hence, when $f_0 = f_1$, we shall have $\alpha = \beta = \frac{90° - \gamma}{2}$, or,

8. *For combined contra-generic cylinders of equal refraction, the plane of greatest positive refraction equally divides the angle between the active plane of the convex cylinder and the axial plane of the concave cylinder; and the plane of greatest negative refraction similarly*

divides the angle between the active plane of the concave cylinder and the axial plane of the convex cylinder.

In case $f_0 > f_1$, then $\beta > \alpha$; or,

9. *When the convex cylinder is stronger than the concave cylinder, the plane of greatest positive refraction will be nearer to the active plane of the convex cylinder, while the plane of greatest negative refraction will be proportionately farther from the active plane of the concave cylinder.*

In case $f_1 > f_0$, then $\alpha > \beta$; or,

10. *When the concave cylinder is stronger than the convex cylinder, the plane of greatest negative refraction will be nearer to the active plane of the concave cylinder, while the plane of greatest positive refraction will be proportionately farther from the active plane of the convex cylinder.*

This is manifest in the diagram.

The values of α and β may be expressed in terms of f_1 f_0 and γ in a similar manner to that shown in the previous theorem, when it can be shown that,

$$\cos \alpha = \sqrt{\frac{1}{2} + \frac{1}{2} \frac{f_0 - f_1 \cos 2\gamma}{\sqrt{f_0^2 - 2f_0f_1 \cos 2\gamma + f_1^2}}} \quad \ldots (VI)$$

This and the transposed equation (36), $\beta = 90° - (\gamma + \alpha)$, suffice to locate the positions of the principal planes of refraction; the angle α being counted from the axis of the *convex* cylinder.

2. *Positions of the Positive and Negative Focal Planes.*

The positions of the positive and negative focal planes will evidently here also be determined by the resultant rays, DM_1 and dm_1, and their correlative intersections with the optic axis at O_1 and O_0.

O_1m_3 will therefore represent one-half the focal line in the positive region behind the lenses, and O_0M_3 one-half the virtual focal line in the negative region before the same.

The ellipses shown in the planes E_1 and E_0 are of the same significance in this as in the preceding combination.

In the plane of greatest positive refraction, DD_1YO_1, we have

$$DY : DD_1 = YO_1 : D_1M_1.$$

Substituting, $DY = O_1o = F_1$ as the positive focal distance,

$$DD_1 = f_1;$$

$$YO_1 = Do = \text{radius} = 1.$$

$$\therefore F_1 = \frac{f_1}{D_1M_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (47)$$

In the parallelogram $D_1V_1M_1Z_1$, the angle between the forces, D_1V_1 and D_1Z_1, is equal to $180° - \gamma$, since $D_1Z_1 \perp Z_1o_1$, and $D_1V_1 \perp A_1o_1$.

$$\therefore D_1M_1 = \sqrt{(D_1Z_1)^2 + (D_1V_1)^2 + 2(D_1Z_1)(D_1V_1)\cos(180° - \gamma)}.$$

In the oblique plane $D_0V_0DV_1D_1$, we find,

$$D_1V_1 : DD_1 = D_0V_0 : DD_0.$$

$$D_0V_0 = \sin \sphericalangle D_0o_0A_0 = \sin \sphericalangle D_1o_1A_1 = \sin \beta.$$

$$DD_0 = f_0.$$

$$\therefore D_1V_1 = \frac{f_1}{f_0} \sin \beta.$$

$$D_1Z_1 = \sin (\sphericalangle Z_1o_1c_1 - \sphericalangle D_1o_1c_1) = \sin (90° - \alpha) = \cos \alpha.$$

Substituting these values in the equation for D_1M_1, equation (47) becomes,

$$F_1 = \frac{f_1}{\sqrt{\cos^2 \alpha + \left(\frac{f_1}{f_0}\right)^2 \sin^2 \beta - 2 \frac{f_1}{f_0} \sin \beta \cos \alpha \cos \gamma}},$$

and which, through equation (36), may be given the form:

$$F_1 = \frac{f_1 f_0}{\sqrt{\frac{1}{2}\left[f_1(f_1 - f_0 \cos 2\gamma) + (f_0 - f_1)\left(f_0 + \sqrt{f_0^2 - 2f_0f_1 \cos 2\gamma + f_1^2}\right)\right]}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (48)$$

Substituting, $\cos 2\gamma = 1 - 2 \sin^2 \gamma$,

$$F_1 = \frac{f_1 f_0}{\sqrt{\frac{(f_0 - f_1)^2}{2} + f_0 f_1 \sin^2 \gamma + (f_0 - f_1)\sqrt{\frac{(f_0 - f_1)^2}{4} + f_0 f_1 \sin^2 \gamma}}}. \quad \text{(VII)}$$

This formula, when reduced for cylinders of equal positive and negative refraction, f_0 being equal to $f_1 = f$, assumes the simple form

$$F_1 = \frac{f}{\sin \gamma} \quad \text{.....} \quad (IX)$$

In the plane of greatest negative refraction, $d_1m_1dO_0X$, we obtain,

$$dX : dd_1 = XO_0 : d_1m_1.$$

Substituting, $dX = O_0o = -F_0$ as the negative focal distance;

$$dd_1 = f_1;$$

$$XO_0 = do = \text{radius} = 1.$$

$$\therefore -F_0 = -\frac{f_1}{d_1m_1}; \quad \text{.....} \quad (49)$$

since d_1m_1 is to be taken negative.

In the parallelogram $d_1v_1m_1z_1$, the angle between the forces, d_1v_1 and d_1z_1, is again $180° - \gamma$; hence,

$$d_1m_1 = \sqrt{(d_1z_1)^2 + (d_1v_1)^2 + 2\,(d_1z_1)\,(d_1v_1)\cos(180° - \gamma)}.$$

In the oblique plane $d_0v_0dv_1d_1$, we find,

$$d_1v_1 : dd_1 = d_0v_0 : dd_0.$$

$$d_0v_0 = \sin(\sphericalangle D_0o_0d_0 - \sphericalangle D_0o_0A_0) = \sin(90° - \sphericalangle D_1o_1A_1)$$

$$= \sin(90° - \beta) = \cos\beta.$$

$$dd_0 = f_0.$$

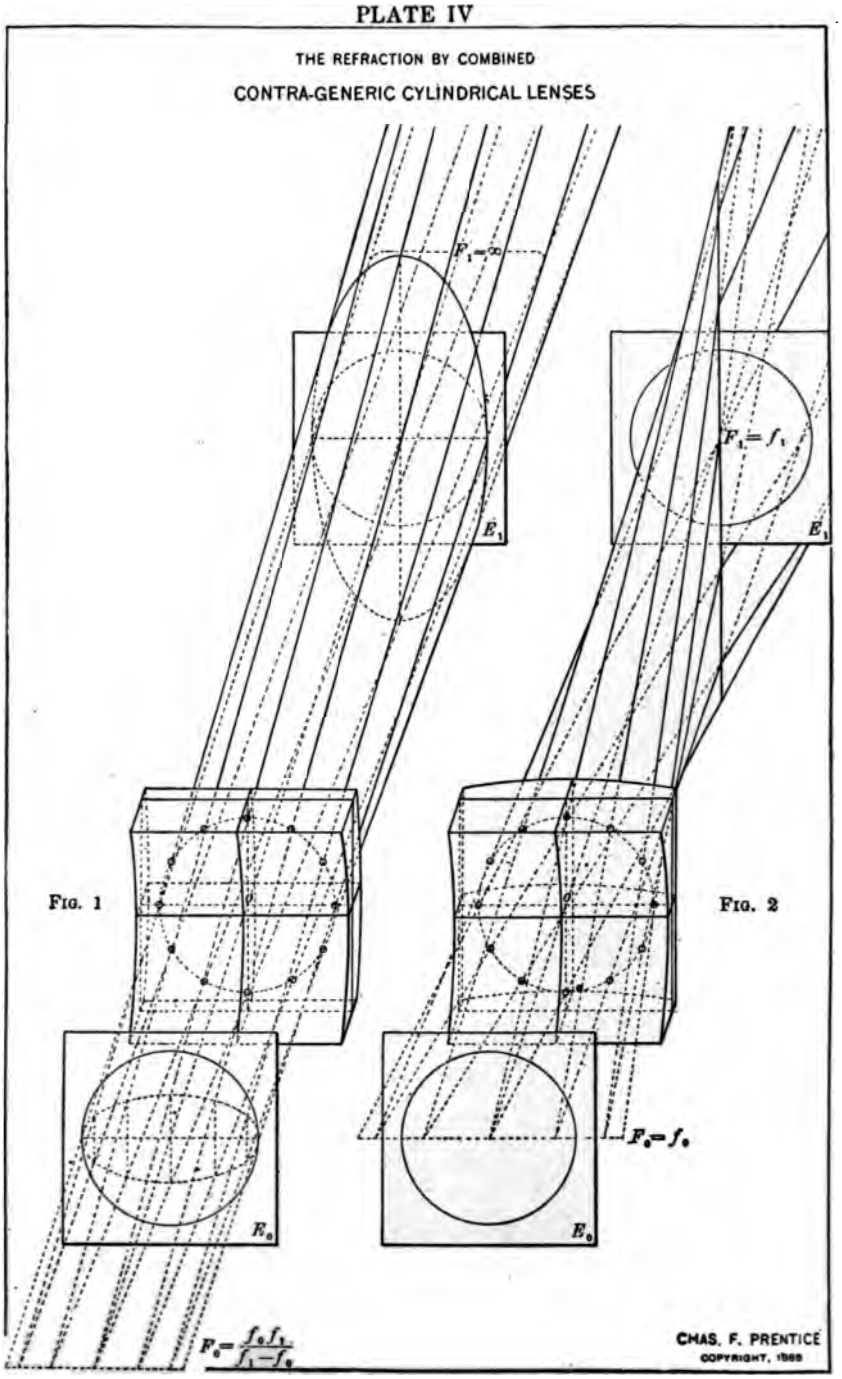
$$\therefore d_1v_1 = \frac{f_1}{f_0}\cos\beta.$$

$$d_1z_1 = \sin \sphericalangle d_1o_1z_1 = \sin\alpha.$$

Substituting these values in the equations for d_1m_1 and (49), we have,

$$-F_0 = -\frac{f_1}{\sqrt{\sin^2\alpha + \left(\frac{f_1}{f_0}\right)^2\cos^2\beta - 2\frac{f_1}{f_0}\sin\alpha\cos\beta\cos\gamma}}$$

PLATE IV


THE REFRACTION BY COMBINED
CONTRA-GENERIC CYLINDRICAL LENSES
$F_1 = \infty$
E_1
$F_1 = f_1$
E_1
FIG. 1
FIG. 2
E_0
E_0
$F_0 = f_0$
$F_0 = \frac{f_0 f_1}{f_1 - f_0}$
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and which, by the aid of equation (36) may be written, $-F_0 =$

$$-\frac{f_1 f_0}{\sqrt{\frac{1}{2}\left[f_1(f_1-f_0\cos 2\gamma)+(f_0-f_1)\left(f_0-\sqrt{f_0^2-2f_0f_1\cos 2\gamma+f_1^2}\right)\right]}}. \quad \ldots\ldots\ldots\ldots (50)$$

$$\therefore -F_0 = -\frac{f_1 f_0}{\sqrt{\frac{(f_0-f_1)^2}{2}+f_0f_1\sin^2\gamma+(f_1-f_0)\sqrt{\frac{(f_0-f_1)^2}{4}+f_0f_1\sin^2\gamma}}}, \quad \ldots\ldots\ldots\ldots (VIII)$$

which differs from the formula given for F_1 merely by a transposition of the elements in the factor before the second radical, and, consequently when reduced to cylinders of equal fraction, also becomes

$$-F_0 = -\frac{f}{\sin\gamma} \quad \ldots\ldots\ldots\ldots (X)$$

The formulæ (IX) and (X) correspond to those which were applied to the Stokes lens.

In reducing the preceding formulæ for given values of cylindric foci, f_0 is to be substituted by the focus of the concave cylinder, and f_1 by the focus of the convex cylinder, both being introduced as positive values.

3. *Relations Between the Positive and Negative Focal Planes.*

As in this combination the cylinders likewise affect each other most when their axes coincide, and least when their axes are diametrically opposed, we may here also fix upon the limits of F_1 and $-F_0$ for $\gamma = 0°$ and $\gamma = 90°$, as in the previous theorem.

When $\gamma = 0°$, or $\cos 2\gamma = +1$, from the equations (48) and (50) we find, for $f_0 > f_1$,

$$F_1 = \frac{f_1 f_0}{\sqrt{\frac{1}{2}[-f_1(f_0-f_1)+(f_0-f_1)(f_0+f_0-f_1)]}} = \frac{f_1 f_0}{f_0-f_1}$$

$$-F_0 = -\frac{f_1 f_0}{\sqrt{\frac{1}{2}[-f_1(f_0-f_1)+(f_0-f_1)(f_0-f_0+f_1)]}} = -\frac{f_1 f_0}{0} = -\infty.$$

$$\therefore F_1 : -F_0 = \frac{f_1 f_0}{f_0 - f_1} : -\infty \ldots\ldots\ldots\ldots (51)$$

For $F_1 = \dfrac{f_1 f_0}{f_0 - f_1}$, we have as the refraction $\dfrac{1}{F_1} = \dfrac{1}{f_1} - \dfrac{1}{f_0}$; consequently,

11. *When the convex cylinder is of greater refraction than the concave cylinder, and their axes are coincident, the positive focal plane will coincide with that focal plane which is defined by the difference of the refractions of the cylinders,* whereas the negative focal plane will be at infinity.*

Placing $\gamma = 0°$, or $\cos 2\gamma = +1$, in the equations (48) and (50), we have, for $f_1 > f_0$,

$$F_1 = \frac{f_1 f_0}{\sqrt{\frac{1}{2}[f_1(f_1 - f_0) - (f_1 - f_0)(f_0 + f_1 - f_0)]}} = \frac{f_1 f_0}{0} = \infty.$$

$$-F_0 = -\frac{f_1 f_0}{\sqrt{\frac{1}{2}[f_1(f_1 - f_0) - (f_1 - f_0)(f_0 - f_1 + f_0)]}} = -\frac{f_1 f_0}{f_1 - f_0}.$$

$$\therefore F_1 : -F_0 = \infty : -\frac{f_1 f_0}{f_1 - f_0} \ldots\ldots\ldots\ldots\ldots (52)$$

For $-F_0 = -\dfrac{f_1 f_0}{f_1 - f_0}$, we have as the refraction

$$-\frac{1}{F_0} = -\left(\frac{1}{f_0} - \frac{1}{f_1}\right) \text{ consequently,}$$

12. *When the concave cylinder is of greater refraction than the convex cylinder, and their axes are coincident, the negative focal plane will coincide with that focal plane which is defined by the difference of the refractions of the cylinders,* whereas the positive focal plane will be at infinity.*

This is shown in Plate IV, Fig. 1.

Introducing $\gamma = 90°$, or $\cos 2\gamma = \cos 180° = -1$ in the equations (48) and (50), we have, for $f_0 \gtreqless f_1$,

* Or the sum of their refractions when taken as positive and negative elements.

$$F_1 = \frac{f_1 f_0}{\sqrt{\frac{1}{2}[f_1(f_1+f_0)+(f_0-f_1)(f_0+f_0+f_1)]}} = \frac{f_1 f_0}{f_0} = f_1.$$

$$-F_0 = -\frac{f_1 f_0}{\sqrt{\frac{1}{2}[f_1(f_1+f_0)+(f_0-f_1)(f_0-f_0-f_1)]}} = -\frac{f_1 f_0}{f_1} = -f_0.$$

$$\therefore F_1 : -F_0 = f_1 : -f_0 \ldots\ldots\ldots\ldots\ldots (53)$$

From which we deduce:

13. *The positive and negative focal planes coincide with their correlative elementary focal planes, when the axes of the contra-generic cylinders are at right angles to each other.*

This is demonstrated in Plate IV, Fig. 2.

Between the limits of 0° and 90°, for $f_0 > f_1$, we have consequently found F_1 to vary between the limits of $\frac{f_1 f_0}{f_0 - f_1}$ and f_1 behind the combined lenses, while F_0 varies between the limits of ∞ and f_0 on the incident side of the same.

The convex cylinder being stronger than the concave cylinder, it is evident when their axes coincide that their combined refraction will be equal to that of a periscopic convex cylinder, since $\frac{1}{F_1} = \frac{1}{f_1} - \frac{1}{f_0}$ in the active plane; and $\frac{1}{F_0} = \frac{1}{\infty} = 0$ in the passive plane.

Between the same limits, when $f_1 > f_0$, F_0 will vary between $\frac{f_1 f_0}{f_1 - f_0}$ and f_0 on the incident side of the combined cylinders, while F_1 varies between ∞ and f_1 behind the same. (See Plate IV.)

In this case, when the axes coincide, it is evident that the resultant refraction will be equal to that of a periscopic concave cylinder, since $\frac{1}{F_0} = -\left(\frac{1}{f_0} - \frac{1}{f_1}\right)$ in the active plane; and $\frac{1}{F_1} = \frac{1}{\infty} = 0$ in the axial plane.

Therefore, with an inequality in the refractive powers of the cylinders, rotation of one of them, from 0° to 90°, will be associated with corresponding changes in the position of the resultant focal planes, between the limits of infinity and the focus of the weaker cylinder on the one side, and between that focal plane which corresponds to the difference of their refractions and the focus of the stronger cylinder on the other. Since in this case the approach of one focal plane is accompanied by a corresponding recession of the other on the opposite side of the lenses, their movements are, as in the previous theorem, in opposite directions.

When the cylinders are of equal refractive power, f_1 being equal to f_0, it follows, from the relation (53), that $F_1 = F_0$, so that between the limits of 0° and 90°, F_1 will vary between infinity and f_1 on the positive side, while F_0 varies between infinity and f_0 on the negative or incident side of the combined cylinders.

Consequently, when the axes coincide, $+F_1 = +\infty$ and $-F_0 = -\infty$. This is evident, since the refractions of equal convex and concave cylinders, under such circumstances, neutralize each other throughout.

By the previous considerations we therefore here also find:

14. *The positive and negative focal planes are conjugate planes, subject to variations of the angle between the axes of the contrageneric cylinders.*

The diagram, Plate III, has been constructed in accordance with the foregoing provisions.

For practical purposes, it will be found more convenient to use the formulæ in the next section.

Dioptral Formulæ for Combined Cylindric Lenses.

As the task of reducing *dioptries* to their focal distances would render calculation by the preceding formulæ somewhat arduous, we may here give the formulæ, expressed in refraction, which will be found especially convenient when applied to combinations of the dioptral system.

Since original publication, these formulæ have been given their simplest possible form. The new formulæ, II*D*, III*D*, VII*D* and VIII*D* are now introduced as sequences to the preceding original formulæ, and whose transformations have been accomplished through convenient substitutions from the equations 54a, 54 and 55.

For the focal distance F_1 we have as the refraction $\frac{1}{F_1} = R_1$, and for

f_1 and f_2, similarly, $\frac{1}{f_1} = r_1$ and $\frac{1}{f_2} = r_2$, which designate the dioptral powers of the cylinders.

By these, and similar substitutions for other foci, we may give the preceding formulæ (I) to (X), the following form:

I. Congeneric Cylinders.

$$\cos a = \sqrt{\frac{1}{2} + \frac{1}{2}\,\frac{r_2 + r_1 \cos 2\gamma}{\sqrt{r_1^2 + 2r_1r_2 \cos 2\gamma + r_2^2}}} \quad \ldots\ldots (ID)$$

$$R_1 = \sqrt{\tfrac{1}{2}(r_1 + r_2)^2 - r_1r_2 \sin^2\gamma + (r_1 + r_2)\sqrt{\tfrac{1}{4}(r_1 + r_2)^2 - r_1r_2 \sin^2\gamma}}$$

$$R_1 = \tfrac{1}{2}\left(r_1 + r_2 + \sqrt{(r_1 + r_2)^2 - 4r_1r_2 \sin^2\gamma}\right) \quad \ldots\ldots (IID)$$

$$R_2 = \sqrt{\tfrac{1}{2}(r_1 + r_2)^2 - r_1r_2 \sin^2\gamma - (r_1 + r_2)\sqrt{\tfrac{1}{4}(r_1 + r_2)^2 - r_1r_2 \sin^2\gamma}}$$

$$R_2 = \tfrac{1}{2}\left(r_1 + r_2 - \sqrt{(r_1 + r_2)^2 - 4r_1r_2 \sin^2\gamma}\right) \quad \ldots\ldots (IIID)$$

To retain the significances of R_1 and R_2, in calculating, r_1 should represent the greater cylindrical refraction.

When the cylinders are of equal power, then $r_1 = r_2 = r$, so that

$$R_1 = r\,(1 + \cos\gamma) \quad \ldots\ldots (IVD)$$

$$R_2 = r(1 - \cos\gamma) \quad \ldots\ldots (VD)$$

II. Contra-Generic Cylinders.

$$\cos a = \sqrt{\frac{1}{2} + \frac{1}{2}\,\frac{r_1 - r_0 \cos 2\gamma}{\sqrt{r_1^2 - 2r_1r_0 \cos 2\gamma + r_0^2}}} \quad \ldots\ldots (VID)$$

$$R_1 = \sqrt{\tfrac{1}{2}(r_1 - r_0)^2 + r_1r_0 \sin^2\gamma + (r_1 - r_0)\sqrt{\tfrac{1}{4}(r_1 - r_0)^2 + r_1r_0 \sin^2\gamma}}$$

$$R_1 = \tfrac{1}{2}\left(r_1 - r_0 + \sqrt{(r_1 - r_0)^2 + 4r_1r_0 \sin^2\gamma}\right) \quad \ldots\ldots (VIID)$$

$$-R_0 = -\sqrt{\tfrac{1}{2}(r_1 - r_0)^2 + r_1r_0 \sin^2\gamma + (r_0 - r_1)\sqrt{\tfrac{1}{4}(r_1 - r_0)^2 + r_1r_0 \sin^2\gamma}}$$

$$-R_0 = -\tfrac{1}{2}\left(r_1 - r_0 - \sqrt{(r_1 - r_0)^2 + 4r_1r_0 \sin^2\gamma}\right) \quad \ldots\ldots (VIIID)$$

When the cylinders are of equal power, then $r_1 = r_0 = r$, hence

$$R_1 = r \sin\gamma \quad \ldots\ldots (IXD)$$

$$-R_0 = -r \sin\gamma \quad \ldots\ldots (XD)$$

If, in (II*D*) and (III*D*), the convex element r_2 be replaced by the concave element $-r_0$, we obtain (VII*D*) and (VIII*D*).

By the aid of the preceding formulæ we may also arrive at the following significant facts.

The formulæ (II*D*) and (III*D*) may be written:

$$R_1^2 = \tfrac{1}{2}(r_1 + r_2)^2 - r_1 r_2 \sin^2\gamma + (r_1 + r_2)\sqrt{\tfrac{1}{4}(r_1 + r_2)^2 - r_1 r_2 \sin^2\gamma},$$

$$R_2^2 = \tfrac{1}{2}(r_1 + r_2)^2 - r_1 r_2 \sin^2\gamma - (r_1 + r_2)\sqrt{\tfrac{1}{4}(r_1 + r_2)^2 - r_1 r_2 \sin^2\gamma},$$

which, by addition, result in the equation,

$$R_1^2 + R_2^2 = (r_1 + r_2)^2 - 2r_1 r_2 \sin^2\gamma.$$

$$\therefore\ (R_1 + R_2)^2 - 2R_1R_2 = (r_1 + r_2)^2 - 2r_1 r_2 \sin^2\gamma.$$
$$\therefore\ (R_1 + R_2)^2 = (r_1 + r_2)^2 - 2r_1 r_2 \sin^2\gamma + 2R_1R_2.$$

Multiplying (II*D*) by (III*D*), we find,

$$2R_1R_2 = 2r_1 r_2 \sin^2\gamma. \quad \ldots\ldots\ldots\ldots\ldots\ldots (54a)$$

$$\therefore\ R_1 + R_2 = r_1 + r_2. \quad \ldots\ldots\ldots\ldots\ldots\ldots (54)$$

From which we conclude:

15. *The sum of the primary and secondary refractions is a constant, being equal to the sum of the elementary refractions for any combination, and all deviations of the axes of two combined congeneric cylinders.*

In the same manner, we obtain from the formulæ (VII*D*) and (VIII*D*),

$$R_1 - R_0 = r_1 - r_0, \quad \ldots\ldots\ldots\ldots\ldots\ldots (55)$$

and therefore here also find,

16. *The sum of the principal positive and negative refractions is a constant, being equal to the sum of the positive and negative elementary refractions for any combination, and all deviations of the axes of two combined contra-generic cylinders.*

As the total inherent refraction always remains the same for any combination, the angle γ merely performs the function of allotting the proportions of refraction, R_1 and R_2, or R_1 and R_0, in the resultant principal planes.

By the equations (54) and (55), calculation may be greatly simplified. R_1 being determined for a specific value of γ, we may readily determine R_2 and R_0, by transforming these equations, as follows:

$$R_2 = r_1 + r_2 - R_1$$

$$-R_0 = r_1 - r_0 - R_1.$$

This is demonstrated in the appended tables.

Tables in Verification of the Dioptral Formulæ.

For Combined Congeneric Cylinders.

Elementary Refractions.	Axial Dev't'n.	Primary Refraction.		Secondary Refraction.		$R_1+R_2=$ r_1+r_2
$r_1 > r_2$	γ	R_1	(*Approx.*)	R_2	(*Approx.*)	
2.5 ⌒ 1.5*D.*	30°	3.75*D.*	3.75*D.*	0.25*D.*	0.25*D.*	4*D.*
2.5 ⌒ 1.5*D.*	45°	3.46	3.5	0.54	0.5	4
2.5 ⌒ 1.5*D.*	60°	3.09	3.	0.91	1.	4

For Combined Contra-Generic Cylinders.

Elementary Refractions.	Axial Dev't'n.	Positive Refraction.		Negative Refraction.		$R_1-R_0=$ r_1-r_0
$r_1 > -r_0$	γ	$+R_1$	(*Approx.*)	$-R_0$	(*Approx.*)	
+ 4 ⌒ — 2.75*D.*	30°	2.397*D.*	+ 2.5*D.*	1.147*D.*	— 1.25*D.*	+ 1.25*D.*
+ 4 ⌒ — 2.75*D.*	45°	3.052	+ 3.	1.802	— 1.75	+ 1.25
+ 4 ⌒ — 2.75*D.*	60°	3.564	+ 3.5	2.314	— 2.25	+ 1.25
$r_1 < -r_0$	γ	$+R_1$	(*Approx.*)	$-R_0$	(*Approx.*)	$R_1-R_0=$ r_1-r_0
+ 2 ⌒ — 2.75*D.*	30°	0.856*D.*	+ 0.75*D.*	1.606*D.*	— 1.5*D.*	— 0.75*D.*
+ 2 ⌒ — 2.75*D.*	45°	1.325	+ 1.25	2.075	— 2.	— 0.75
+ 2 ⌒ — 2.75*D.*	60°	1.690	+ 1.75	2.440	— 2.5	— 0.75

Sphero-Cylindric Equivalence of Combined Cylindric Lenses.

Since, for any combination of cylinders, the principal planes of refraction are at right angles to each other for all values of γ, there can be no reasonable doubt, owing to the provisions made at the opening of this demonstration, as to the equivalence of a sphero-cylindric lens to one composed of combined cylinders. However, as the use of such lenses is at present confined to the correction of errors of refraction in the human eye, it is evident, from the movements of the eye behind the fixed lens, that the visual axis cannot at all times coincide with the optic axis of the lens chosen; therefore, in those instances where substitution of one form of lens for the other proves to be unsatisfactory, the cause may be seemingly explained by a possible difference becoming manifest for the more peripheral incident rays, though these be equally distant from the optic center of each lens. In other words, the available field in the one may be greater or less than in the other; yet even this would probably only be appreciable in lenses of extreme curvature, and possibly in combinations where the cylinders differ widely in power.

To substitute a sphero-cylindric lens for combined cylinders is a proposition which merely demands that the focal interval should be the same, at the same distance from the principal plane, at the optic center, for each of the compound lenses. The distances F_1 and F_2 being determined for any angular deviation, γ, of the axes, in a combination of congeneric cylinders, for instance, the substitution is accomplished by making a sphero-cylindric lens in which the focus of the spheric element is equal to F_2, and of the cylindric element is equal to $\frac{F_1F_2}{F_2 - F_1}$, or, if expressed in refraction, $\frac{1}{F_1} - \frac{1}{F_2}$ sph. $= \frac{1}{F}$ cyl. $= \frac{1}{F_c}$.

Should it be desired to place the primary and secondary planes of the sphero-cylindric lens so as to coincide with those resulting from a combination of two definitely placed congeneric cylinders, it will be necessary to refer to the formula (I) and to the laws 2 and 3.

Comparing the sphero-cylindric equivalent with its corresponding rotating cylinders, reference being had to Plate II, Figs. 2 and 3, we find a decrease in the angle, γ, from 90° to 0° to effect a corresponding decrease in the spheric element, F_2, from the focus f_2 to ∞; this being associated with a cylindric element of the focus F_c, which constantly increases from the focus $\frac{f_1f_2}{f_2 - f_1}$ to $\frac{f_1f_2}{f_2 + f_1}$. In other words, a gradually decreasing potency of the spheric refraction $\frac{1}{F_2}$ from $\frac{1}{f_2}$ to $\frac{1}{\infty} = 0$, gives way to a proportionately increasing cylindric refraction $\frac{1}{F_c}$, from $\frac{1}{f_1} - \frac{1}{f_2}$ to $\frac{1}{f_1} + \frac{1}{f_2}$. As an instance, if $f_1 = f_2 = f$, $\frac{1}{F_c}$ will increase from $\frac{1}{f_1} - \frac{1}{f_2} = 0$ to $\frac{2}{f}$, or twice the refraction of

either cylinder. In this case, all successive values of cylindric refraction will, therefore, be inherent between 0 and $\frac{2}{f}$.

Should a means be devised to suppress the spheric element for each successive value of γ, the remaining varying cylindric element being thus rendered available for measuring corresponding degrees of astigmatism in the eye, the formulæ here advanced would prove of service in obtaining the graduations upon the rotating scale of such an instrument.

While there are few cases of astigmation which demand correction by combined cylinders, we may nevertheless be permitted to passingly allude to certain methods of procedure in such instances. We shall confine the subject to congeneric cylinders. In a case of astigmatism which has been found to be corrected by two cylinders combined under the angle γ, the lenses should be withdrawn from the trial frame, when they are to be superposed with their plane surfaces in contact; and in such manner as to facilitate their being rigidly held in the required position for γ.

The positions of the principal planes of refraction may then be estimated for this fixed combination, the same as if it were a single lens, though without regard to the exact nature of the elements constituting it. The powers of the principal planes of refraction will be revealed by neutralizing with the lenses from the trial case. The spheric and cylindric elements thus determined are then to be substituted in the trial frame, when rotation of the cylinder will lead to that position of it which produces the best acuteness of vision. The spheric and cylindric elements will probably then also bear of further modification, in case any error may have been made at the outset. In lieu of this practical method, recourse must be had to the formulæ.

It having been shown that successive changes in the angle γ are associated with corresponding changes in F_1 and F_2, the above substitution would indeed seem advisable, since the present appliances for grinding bicylindric lenses are not constructed with sufficient precision to enable opticians to fix the relative positions of the cylinders beyond mere approximation.

As an illustration, let us select two congeneric cylinders of equal foci, say 20 inches, combined under the angle $\gamma = 60°$. Introducing these values in the formulæ (IV) and (V), we find,

$$F_1 = \frac{20}{1 + \cos 60^\circ} = \frac{20}{1 + 0.5} = 13.33,$$

$$F_2 = \frac{20}{1 - \cos 60^\circ} = \frac{20}{1 - 0.5} = 40.$$

We then obtain the cylindric refraction $\frac{1}{F_c}$, for the desired sphero-cylindric equivalent, from the equation,

$$\frac{1}{F_1} - \frac{1}{F_2} = \frac{1}{F_c}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (56)$$

Substituting herein the calculated values for F_1 and F_2 gives,

$$\frac{1}{13.33} - \frac{1}{40} = \frac{1}{F_c} = \frac{1}{19.99} = \frac{1}{20} \text{ (nearly)}.$$

$\frac{1}{F_2} = \frac{1}{40}$ being the spherical element, we therefore have the sphero-cylindric equivalent,

$$\frac{1}{40} \text{ sph.} \frown \frac{1}{20} \text{ cyl.}$$

as an available substitute for the cylindro-cylindric lens,

$$\frac{1}{20} \text{ cyl. axis } 0^\circ \frown \frac{1}{20} \text{ cyl. axis } 60^\circ$$

without regard to a definite position of these lenses before the eye.

By way of comparison, allowing the optician to make an error of apparently so small an amount as 2°, in producing the same cylindro-cylindric lens, we obtain, by introducing $\gamma = 62^\circ$ in the same formulæ,

$$F_1 = \frac{20}{1 + \cos 62^\circ} = \frac{20}{1 + 0.469} = \frac{20}{1.47} = 13.61,$$

$$F_2 = \frac{20}{1 - \cos 62^\circ} = \frac{20}{1 - 0.47} = \frac{20}{0.53} = 37.73.$$

Substituting these values in the equation (56), we have,

$$\frac{1}{13.61} - \frac{1}{37.73} = \frac{1}{F_c} = \frac{1}{21.29},$$

from which we obtain the sphero-cylindric lens,

$$\frac{1}{37.73} \text{ sph.} \subset \frac{1}{21.29} \text{ cyl.}$$

Had the optician been required to make a sphero-cylindric lens $\frac{1}{40}$ sph. $\subset \frac{1}{20}$ cyl., his execution of it presenting such discrepancies as $\frac{1}{37.73}$ sph. $\subset \frac{1}{21.29}$ cyl., would certainly be rejected as being unsatisfactory, on account of the notable difference of 2.27 inches in the focal distance of the spheric element.

On the other hand, instances are likely to occur in which it will be impossible, by the advanced method of neutralization to accurately arrive at the sphero-cylindric equivalent.

Since $\frac{1}{20}$ cyl., axis 0° $\subset \frac{1}{20}$ cyl., axis 62° $= \frac{1}{37.73}$ sph. $\subset \frac{1}{21.43}$ cyl., we should evidently be unable to accurately neutralize such spheric and cylindric elements by any of the lenses from the trial case.

In those instances, therefore, where satisfactory neutralization of the principal planes of refraction in a pair of combined cylinders cannot be attained, the cylindro-cylindric lens will have to be chosen, again under the proviso, however, of a faultless mechanical execution. However, as in most instances a sphero-cylindrical equivalent will be available, *we are to suspect error in our estimate of the refraction of an eye which seems to demand cylinders combined under acute or obtuse angles.*

The following is a case in point:

A cylindro-cylindric lens $-\frac{1}{40}$ cyl. axis 0° $\frown$ $-\frac{1}{40}$ cyl. axis 70°

had been prescribed for Mr. G. B. O., of New York, by his oculist in Philadelphia, in 1880–'1. With this lens the vision equalled $\frac{6}{6}$ for the left eye.

In this instance the sphero-cylindric equivalent was obtained as follows:

The lenses being congeneric concave cylinders of equal refraction, by the formulæ (IV) and (V), for $f = 40$ and $\gamma = 70°$, we have,

$$F_1 = \frac{40}{1 + 0.34202} = 29.806 = 30,$$

$$F_2 = \frac{40}{1 - 0.34202} = 60.79 = 60,$$

it being admissible to neglect the fractions for such focal distances.

By law 2, we find the position of the cylindric axis equal $\frac{\gamma}{2} = 35°$,

and consequently the sphero-cylindric equivalent,

$$-\frac{1}{60} \text{ sph.} \frown -\frac{1}{60} \text{ cyl., axis } 35°.$$

This lens was substituted with the knowledge and to the entire satisfaction of the patient.

It is, therefore, obvious that the meridian (125°) of greatest refraction in the eye had not been disclosed by the diagnosis.

The weak spheric element in the substituted lens, while being an appreciable factor to the patient, might easily have been overlooked by the practitioner.

In similar cases, the advanced formulæ should prove of value in fixing upon the true state of the refraction.

CHRONOLOGICAL DATA

Abstracted from *The Keystone Magazine of Optometry*, July 15, 1915

Charles F. Prentice was born in Brooklyn, N. Y., June 24, 1854. In order that he might ultimately be of expert assistance to his father, James Prentice, of London, England, a noted optician established in New York in 1842, he was educated in the department of mechanical engineering at the Royal Polytechnikum, Karlsruhe, Germany. He became his father's business partner in 1882, and in 1888 succeeded him in the conduct of a lucrative optical business, which thereafter gradually developed into a strictly professional one. Through close application and original research in optics, he soon became known as the author of many highly technical articles in leading ophthalmic journals in this country, a fact which also gave him immediate recognition among ophthalmologists of international eminence, such as Helmholtz, Knapp, Noyes, Wood, Burnett and others. From among his numerous scientific contributions of characteristic originality the most important are here given in the order of their publication, as follows:

1884. "*Prentice's Class Model,*" demonstrating refraction and accommodation of the eye. Now used in the department of physics, Columbia University.

1886. "*Ophthalmic Lenses.*" The first published treatise on the subject, and under the name by which all lenses used by the eye practitioner have since been known.

1888. "*Dioptric Formulæ for Combined Cylindrical Lenses.*" This being the first and only solution of the problem to disclose the sixteen laws which he discovered to apply to such combinations.

1890. "*A Metric System of Numbering and Measuring Prisms.*" A system for which he named the unit, the prism-dioptry, and discovered the law of decentration. This system was promptly adopted by leading American manufacturers, and is now in universal use in this country.

1895. "*Why Strong Contra-generic Lenses of Equal Power Fail to Neutralize Each Other.*" The reason disclosed led the author to suggest that all convex lenses shall be made weaker than standard concave lenses, in order to effect neutralization. Mr. Prentice has since demonstrated that two lenses which effectually neutralize each other, through actual contact of their contiguous surfaces, are of the same vertex-refraction.

1895. "*The Iris, as Diaphragm and Photostat.*" The first lucid explanation of the physical reason why sub-decimals of the lens dioptry prove efficacious.

1897. "*The Typoscope.*" A simple device used in conjunction with spectacles for increasing the visual acuteness of amblyopes, and of which Dr. H. Knapp said: "I am delighted with the typoscope. It rests in sound physiological principles and will benefit many people."

1914. He published several articles on the "*Chiasmal Image,*" an original conception which brought forth a single dissenting criticism, but which he was able to effectually refute.

1915. "*Vertex-Refraction in Its True Aspect.*" A mathematical demonstration proving the value of the principle of neutralization, and which twenty years earlier he had advocated should also be extended to lenses of high power.

1915. "*The Fundamentals of Achromatism.*" A novel description of achromatism, in which the individual courses of the color-rays are visibly traced through the refractive media and, upon emergence from an achromatic prism, are shown to become parallel to produce white light; whereas, they are collectively superposed at the focus of an achromatic lens to produce it.

His literary treatment of the prism-dioptry was pronounced "classic," while it has also been authoritatively stated that, "such work as Mr. Prentice's puts us on the proper footing in the scientific world."

He also originated the optical terms, prism-dioptry, contra-generic, dioptral, photostat, typoscope, chiasmal image and contrameniscus. Besides, he is not less conspicuous as an altruist, since he has never profited financially from any of his contributions to science.

ABSTRACTS FROM PRESS REVIEWS

Of Previous Publications, by Charles F. Prentice

"To Prentice alone belongs the credit, let us not deprive him of the honor."—*Medical News,* Philadelphia, 1891.

"All that is necessary is to remember the law laid down by Prentice."—*A Text Book of Ophthalmology,* by W. F. Norris, M. D., and Chas. A. Oliver, M. D., Profs. of Oph., University of Penn.

"With what seems like a flash of genius, Prentice has put forth the new system."—Annual meeting of the American Medical Association, Washington, D. C., 1891.

"Such work as Prentice's puts us on the proper footing in the scientific world."—S. M. Burnett, M. D., Prof. of Oph. Georgetown University, Washington, D. C., 1888.

"We are indebted to Prentice for the new system of prism-measurement" (lit. trans.).—*Les Annales d'Oculistique,* Paris, 1892.

"This method was first suggested by Prentice, who has gone very thoroughly into the mathematics of the subject."—*The Ophthalmic Review,* London, 1891.

"The exhaustive work of Prentice on cylindrical lenses, is a solution of one of the most difficult problems in optics."—*The Ophthalmic Review,* London, 1888.

"Nowhere is there to be found so complete an explanation as is contained in Prentice's work."—*The Archives of Ophthalmology,* New York, 1888.

"Prentice's prism-dioptral system is infinitely superior to the old method."—*Journal of Ophthalmology, Otology, and Laryngology,* New York, 1892.

"Prentice's well known technical skill and knowledge of mathematical optics are a sufficient guaranty. . . ." (lit. trans.).—*Archiv für Augenheilkunde,* Berlin, 1890.

"Prentice's papers on the prism-dioptry are classic."—*Annals of Ophthalmology,* St. Louis, 1901.

"In physiologic optics one must mention the works of Prentice, Dennett and Knapp, of New York."—The Progress of Ophthalmology, *Journal of the American Medical Association,* Chicago, 1899.